FOREWORD

I am indebted to my two nephews, mas[illegible] B[illegible] Kumar Sen, B Sc, and Dhire[illegible] Kumar Sen for the production of this Solu[illegible] Half the credit is theirs and half is mine O[illegible] to strictly correct, I may say with the [illegible] hev, more than half to the two boys [illegible] ending

Allahabad
The 29th May, 1915. } J. N. Sen.

SOLUTION OF EXERCISES

IN

HALL & STEVEN'S GEOMETRY, PART I.

Page 13.

1. We know from definition

When the sum of two angles is two right angles or $180°$, ...h of two angles is said to be supplement of the other.

Hence,

Supplement of one-half of a right angle is 2 rt. $\angle^s - \frac{1}{2}$ rt. ... or $\frac{3}{2}$ rt. $\angle^s$, or three-halves of a right angle, or $\frac{3}{2} \times 90°$, ... $135°$.

Supplement of four-thirds of a right angle is 2 rt. $\angle^s$ ... $\frac{4}{3}$ rt. $\angle^s$, or $\frac{2}{3}$ rt. $\angle^s$, or two-thirds of a right angle, or $\frac{2}{3}$ ... $90°$, or $60°$

Supplement of $46°$ is $180° - 46°$, or $134°$.

Supplement of $149°$ is $180° - 149°$, or $31°$.

Supplement of $83°$ is $180° - 83°$, or $97°$.

Supplement of $101°. 15'$ is $180° - 101°. 15'$, or $78°. 45'$.

2. We know from definition

When the sum of two angles is one right angle or $90°$, ...ch of two angles is said to be complement of the other.

Hence,

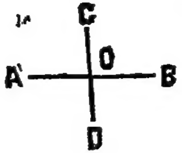
Complement of two-fifths of a right angle is 1 rt. $\angle -$ ... rt. $\angle$, or $\frac{3}{5}$ rt. $\angle$, or three fifths of a right angle, or $\frac{3}{5} \times 90°$, ... $54°$.

Complement of $27°$ is $90° - 27°$, or $63°$.

Complement of $38°. 16'$ is $90° - 38°. 16'$, or $51°. 44'$.

Complement of $41°. 29'. 30''$ is $90° - 41°. 29'. 30''$, or $48°$. ...$'. 30''$.

3. Let AB, CD be any two straight lines intersecting at O, and let AOC be a right angle.



It is required to prove that the other three angles COB, BOD and AOD are also right angles.

Proof.—Because OC stands on the straight line AB at O,

∴ the $\angle^s$ AOC and COB are together equal to 2 rt. $\angle^s$; (Theor. 1)

But the $\angle$ AOC is given a rt. $\angle$.
Hence, the $\angle$ COB is also a rt. $\angle$.

Again because AO stands on the straight line DC at O

∴ the $\angle^s$ AOC and AOD are together equal to 2 rt. $\angle^s$. (Theor. 1)

But the $\angle$ AOC is given a rt. $\angle$
Hence, the $\angle$ AOD is also a rt. $\angle$.

Again because DO stands on the straight line AB at O.

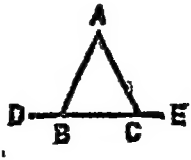
∴ the $\angle^s$ AOD and DOB are together equal to 2 rt. $\angle^s$. (Theor. 1)

But the $\angle$ AOD is proved to be a rt. $\angle$.
Hence, the $\angle$ DOB is also a rt. $\angle$.

∴ each of the three angles COB, AOD and DOB is a rt. $\angle$.

Q. E. D.

4. Let ABC be a triangle in which the angles ABC and ACB are given equal. The side BC is produced both ways to the points D and E, as shown in the diagram.



It is required to prove that the angles ABD and ACE equal.

Proof.—Because AB stands on DE at B,

∴ the $\angle^s$ ABD and ABC are together equal to two ht angles. (Theor. 1)

Also because AC stands on DE at C,

∴ the $\angle^s$ ACE and ACB are together equal to two right gles. (Theor. 1)

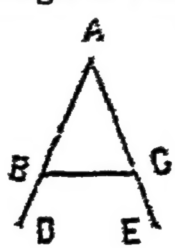
∴ the $\angle^s$ ABD and ABC together = the $\angle^s$ ACE and CB together.

But the $\angle$ ABC = the $\angle$ ACB (by hypothesis)

∴ the $\angle$ ABD = the $\angle$ ACE.

Q. E. D.

5. Let ABC be a triangle in which the angles ABC and CB are given equal. The side AB is produced beyond B to any point D and AC is roduced beyond C to any point E.



It is required to prove that the angles BCE and CBD are equal.

Proof.—Because BC stands on AD at B,

∴ the $\angle^s$ ABC and CBD are together equal to two right angles. (Theor. 1)

Again because BC stands on AE at C,

∴ the $\angle^s$ ACB and BCE are together equal to two right angles. (Theor. 1)

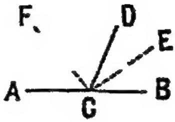
∴ the $\angle$ ABC and CBD together = the $\angle^s$ ACB and BCE together.

But the ∠ ACB = the ∠ ABC (by hypothesis)

∴ the ∠ CBD = the ∠ BCE.

Q. E. D.

6 Let the straight line CD stand on another straight line AB at C making the adjacent angles BCD and DCA. CE bisects the angle DCB and CF bisects the angle DCA.



It is required to prove that the angle FCE is a right angle.

Proof.—Because CD stands on AB at C,

∴ the ∠ BCD and the ∠ ACD together = 2 rt $\angle^s$

(Theor. 1)

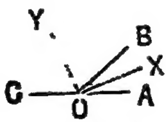
∴ ½ the ∠ BCD and ½ the ∠ ACD together = one rt ∠.

But the ∠ DCE is half of the ∠ BCD and the ∠ DCF is half of the ∠ ACD (by hypothesis)

∴ the ∠ DCE and the ∠ DCF together = one rt ∠ i e, the ∠ FCE is a rt. ∠.

Q E. D.

7. Let the straight line BO stand on AC at O making the adjacent angles BOA and BOC OX is bisector of the angle BOA and OY is bisector of the angle BOC



It is required to prove that the $\angle^s$ AOX and COY are complementary.

Proof.—Because OB stands on CA at O,

∴ the ∠ BOA and the ∠ BOC together = 2 rt. $\angle^s$

(Theor. 1)

$\therefore$ $\frac{1}{2}$ the $\angle$ BOA and $\frac{1}{2}$ the $\angle$ BOC = one rt. $\angle$.

But the $\angle$ AOX is half of the $\angle$ BOA and the $\angle$ COY half of the $\angle$ BOC (by hypothesis)

$\therefore$ the $\angle$ AOX and the $\angle$ COY together = one rt. $\angle$

$\therefore$, the $\angle^s$ AOX and COY are complementary. (from definition)

Q. E. D.

8. (See fig. in Ex. 7)

Let BO stand on the straight line AC at O making the adjacent angles BOA and BOC. OX is bisector of the angle AOB and OY is bisector of the angle BOC.

It is required to prove that the angles BOX and COX are supplementary, and also that the angles AOY and BOY are supplementary.

Proof.—Because OX stands on AC,

$\therefore$ the $\angle^s$ AOX and XOC are together equal to two right angles. (Theor. 1)

But the $\angle$ AOX = the $\angle$ BOX (by hypothesis)

$\therefore$ the $\angle^s$ BOX and XOC together = 2 rt. $\angle^s$

i.e., the $\angle^s$ BOX and BOX are supplementary (from definition)

Again because OY stands on CA,

$\therefore$ the $\angle^s$ COY and YOA together = 2 rt. $\angle^s$

(Theor. 1)

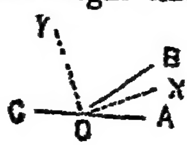
But the $\angle$ COY = the $\angle$ BOY (by hypothesis)

$\therefore$ the $\angle^s$ BOY and YOA together = 2 rt. $\angle^s$

i.e., the $\angle^s$ BOY and YOA are supplementary.

Q. E. D.

9. Let BO stand on the straight line AC at O making the adjacent angles BOA and BOC. OX is bisector of the angle AOB and OY is bisector of the angle BOC.



Let the angle AOB be 35°.

It is required to find the angle COY.

Because BO stands on CA

$\therefore$ the $\angle^s$ BOC and BOA together $=2$ rt. $\angle^s$ (Theor. 1)

And because the $\angle$ AOB$=35°$,

$\therefore$ the $\angle$ BOC$=180°-35°=145°$.

But the angle COY is half of the $\angle$ BOC (by hypothesis)

$\therefore$ the $\angle$ COY$=\frac{1}{2}$ of $145°=72°, 30'$.

Page 15.

1. Since the minute-hand of a clock makes one complete revolution in one hour or 60 minutes, it turns through 4 rt. $\angle^s$ in 60 minutes

i. e. in 60 minutes the minute-hand turns through 360°.

$\therefore$ in one minute the minute-hand will turn through $\frac{360}{60}°$, or 6°.

(*i*) $\therefore$ in 5 minutes the minute-hand will turn through $6°\times 5$, or 30°.

(*ii*) in 21 minutes the minute-hand will turn through $6°\times 21$, or 126°.

(*iii*) in $43\frac{1}{2}$ minutes or $\frac{87}{2}$ minutes the minute-hand will turn through $6°\times\frac{87}{2}$, or 261°

(*iv*) in 14 minutes 10 seconds or $\frac{85}{6}$ minutes the minute-hand will turn through $6°\times\frac{85}{6}$, or 85°.

The minute-hand turns through 6° in one minute.

(*v*) the minute hand will turn through 66° in $\frac{66}{6}$ minutes, or 11 minutes.

(*vi*) the minute-hand will turn through 222° in $\frac{222}{6}$ minutes, or 37 minutes.

2. (See fig. in Ex. 1.)

Since, the hour-hand of a clock makes one complete revolution in 12 hours, it turns through 4 rt $\angle^{s}$ in 12 hours.

i. e. in 12 hours the hour-hand turns through 360°.

$\therefore$ in 1 hour the hour-hand will turn through $\frac{360°}{12}$, or 30°.

(*i*) $\therefore$ in 3 hours 45 minutes, or $\frac{15}{4}$ hours, the hour-hand will turn through $30° \times \frac{15}{4}$, or $\frac{225}{2}°$, or 112° 30′.

(*ii*) in 5 hours 10 minutes or $\frac{31}{6}$ hours the hour-hand will turn through $30° \times \frac{31}{6}$, or 155°.

The time taken by the hour-hand in turning through 30° is one hour.

$\therefore$ the time taken by the hour-hand in turning through

$$172\tfrac{1}{2}°, \text{ or } \frac{345°}{2} \text{ will be } \frac{\frac{345°}{2}}{30°}, \text{ or } \frac{345}{2 \times 30} \text{ hours, or } \frac{23}{4} \text{ hours,}$$

or 5 hours 45 minutes

3. Since the earth makes one complete revolution in 24 hours about its axis, it turns through 4 rt. $\angle^{s}$ in 24 hrs.

i. e. in 24 hours the earth turns through 360°.

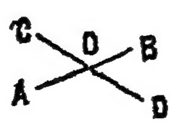
$\therefore$ in one hour the earth will turn through $\frac{360°}{24}$, or 15°.

(*i*) $\therefore$ in 3 hours 20 minutes, or $\frac{10}{3}$ hours the earth will turn through $15° \times \frac{10}{3}$, or 50°.

The time taken by the earth in turning through 15° is one hour.

(*ii*) $\therefore$ the time taken by the earth in turning through 130° will be $\frac{130°}{15°}$, or $\frac{26}{3}$ hours, or 8 hours 40 minutes.

4. Let the straight lines AB and AD cut one another at the point O.



(*i*) If the angle AOC be 35°, it is required to find the value of each of the angles COB, BOD, DOA without measurement.

Because the straight line CO stands on AB

$\therefore$ the $\angle^s$ AOC and COB together $=2$ rt. $\angle^s$, or $180°$.

(Theor. 1).

But the $\angle$ AOC $=35°$ (given)

$\therefore$ the $\angle$ COB $=180°-35°$, or $145°$.

Because AB and CD cut one another at O

$\therefore$ the $\angle$ COB $=$ the vertically opposite $\angle$ DOA

(Theor. 3)

But the $\angle$ COB $=145°$ (proved)

$\therefore$ the $\angle$ DOA $=145°$.

Also, the $\angle$ AOC $=$ the vertically opposite $\angle$ BOD

(Theor. 3)

But the $\angle$ AOC $=35°$ (given)

$\therefore$ the $\angle$ BOD $=35°$

(ii) If the $\angle^s$ COB and AOD together be $250°$, it is required to find each of the $\angle^s$ COA, BOD.

Because AB and CD cut one another at O

$\therefore$ the $\angle^s$ AOC, COB, BOD and DOA together $=4$ rt. $\angle^s$, or $360°$. (Cor. 1 Theor. 1)

But the $\angle^s$ BOB and AOD together $=250°$ (given)

$\therefore$ the $\angle^s$ COA and BOD together $=360°-250°$, or $110°$.

But the $\angle$ AOC $=$ vertically opposite $\angle$ BOD (Theor. 3)

$\therefore$ each of the $\angle^s$ AOC, BOD $=\frac{1}{2}$ of $110°$, or $55°$.

(iii) If the $\angle^s$ AOC, COB, BOD together make up $274°$, it is required to find each of the $\angle^s$ AOC, COB, BOD, DOA.

Because CO stands on AB

$\therefore$ the $\angle^s$ AOC and COB together $= 2$ rt. $\angle^s$, or $180°$ (Theor. 1)

But the $\angle^s$ AOC, COB. BOD together $= 274°$ (given)

$\therefore$ the $\angle$ BOD $= 274° - 180°$, or $94°$.

Because DO stands on AB

$\therefore$ the $\angle^s$ BOD and DOA together $= 2$ rt. $\angle^s$, or $180°$ (Theor. 1)

But the $\angle$ BOD $= 94°$ (proved)

$\therefore$ the $\angle$ DOA $= 180° - 94°$, or $86°$.

Because AB and CD cut one another at O

$\therefore$ the $\angle$ AOC $=$ vertically opposite $\angle$ BOD (Theor. 3)

But the $\angle$ BOD $= 94°$ (proved)

$\therefore$ the $\angle$ BOC $= 94°$

Also the $\angle$ BOC $=$ vertically opposite $\angle$ AOD (Theor. 3)

But the $\angle$ DOA $= 86°$ (proved)

$\therefore$ the $\angle$ AOC $= 80°$.

5. (See fig. in Ex 4).

Let AB be a straight line and let O be any point in it from which two straight lines OC and OD are drawn on opposite sides of AB such that the angles COB and AOD are equal.

It is required to prove that OC and OD are in the same straight line.

Proof.—Because OC stands on AB

$\therefore$ the $\angle^s$ AOC and COB together $= 2$ rt. $\angle^s$ (Theor. 1)

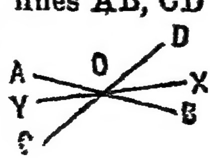
But the $\angle$ COB $=$ the $\angle$ AOD (by hypothesis)

$\therefore$ the $\angle^s$ AOC and AOD together $= 2$ rt. $\angle^s$

$\therefore$ OC and OD are in the same straight line (Theor. 2)

Q. E D.

6. Let two stright lines AB, CD cross one another at O, and let OX be the bisector of the angle BOD. Produce XO beyond O to any point Y.



It is required to prove that OY bisects the angle AOC.

Proof —Because AB and YX cut at O.

∴ the ∠ BOX = vertically opposite ∠ AOY (Theor. 3)

Again because CD and YX cut at O.

∴ the ∠ COY = vertically opposite ∠ DOX (Theor. 3)

But the ∠ DOX = the ∠ BOX (by hypothesis)

∴ the ∠ AOY = the ∠ COY.

i. e. the ∠ AOC is bisected by OY. Q. E. D.

7. (See fig. in Ex. 6)

Let two stright lines AB CD intersect at O, let OX be the bisector of the angle BOD, and OY the bisector of the ∠ AOC.

It is required to prove that OX and OY are in the same straight line.

Proof—Because AB and CD cut at O.

∴ the ∠ AOC = vertically opposite ∠ BOD (Theor. 3)

But the ∠ AOY = the ∠ COY (by hypothesis) = $\frac{1}{2}$ the ∠ AOC.

Also, the ∠ BOX = the ∠ DOX (by hypothesis) = $\frac{1}{2}$ the ∠ BOD.

But the ∠ AOC = the ∠ BOD (proved)

∴ the ∠ AOY = the ∠ BOX.

Because OX stands on AB

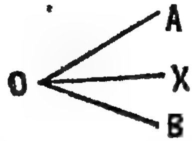
∴ the $\angle^s$ BOX aad XOA together = 2 rt $\angle^s$ (Theor. 1)

But the ∠ BOX = the ∠ AOY (proved)

∴ the $\angle^s$ AOY and XOA together = 2 rt. $\angle^s$

∴ OX and OY are in the same straight line. (Theor. 2)
Q. E. D.

8. Let AOB be a given angle and let OX be the bisector of the angle AOB.



It is required to show that, by folding the diagram about OX, OA may be made to coincide with OB.

In folding the diagram about OX we make the ∠ AOX fall upon the ∠ BOX.

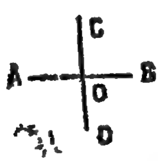
But the ∠ AOX = the ∠ BOX (by hypothesis)

∴ OA will fall upon OB.

(*i*) If the ∠ AOX is greater than the ∠ XOB, OA will fall outside the ∠ XOB with regard to OB, so that OX and OA will be on opposite sides of OB.

(*ii*) If the ∠ AOX is less than the ∠ XOB, OA will fall within the ∠ XOB with regard to OB, so that OX and OA will be on the same side of OB.

9. Let AB and CD cut one another at right angles at O.



It is required to show that, by folding the figure about AB, OC may be made to fall along OD.

In folding the figure about AB, we make the straight angle on the left side of AB fall on the straight angle on the right side of AB.

But the ∠ AOC = the ∠ AOD (being rt. ∠s)

∴ OC will fall along OD.

10 (See fig in Ex. 9.)

Let AB be a straight line drawn on paper and let O be any point in it about which the straight line AB is so folded that OA falls along OB Let COD be the crease left on the paper

It is required to prove that COD is perpendicular to AB.

Proof.—Because the ∠ AOC falls upon the ∠ COB such that OA falls along OB.

∴ the ∠ AOC = the ∠ COB.

But these are adjacent angles.

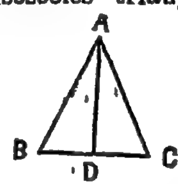
∴ CO is perpendicular to AB.

or, CD is perpendicular to AB.

Q. E. D.

Page 19.

1. Let ABC be an isosceles triangle and let AD be the bisector of the vertical angle BAC meeting BC in D.



(i) It is required to prove that AD bisects the base BC.

Proof.—In the two $\triangle^s$ BAD and ACD

Because
- the side BA = the side AC (being sides of an isosceles triangle)
- the side DA is common to both.
- and the included ∠ BAD = the included ∠ CAD (by hypothesis)

∴ two triangles BAD and ACD are equal in all respects.

(Theor. 4)

so that $BD=DC$.

i. e BC is bisected at D.

(ii) It is required to prove that AD is perpendicular to BC.

Proof.—In the two $\triangle^s$ BAD and ADC.

Because $\begin{cases} \text{the side } AB = \text{the side } AC \text{ (being sides of an isosceles triangle)} \\ \text{the side } AD \text{ is common to both} \\ \text{and the included } \angle BAD = \text{the included } \angle DAC \\ \quad \text{(by hypothesis)} \end{cases}$

$\therefore$ the two $\triangle^s$ are equal in all respects. (Theor. 4)

so that, the $\angle ADB =$ the $\angle ADC$

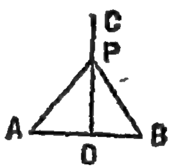
and these being adjacent angles, each is a rt. $\angle$,

(From definition)

$\therefore$ AD is perpendicular to BC.

Q. E. D.

2. Let AB be a straight line, and O its middle point. Let OC be perpendicular to AB at O and let P be any point in OC. Join PA and PB.



It is required to prove that PA and PB are equal.

Proof.—In the two $\triangle^s$ POA and POB

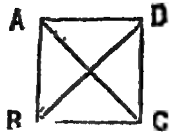
Because $\begin{cases} PO \text{ is common to both} \\ AO = OB \text{ (by hypothesis)} \\ \text{and the included } \angle POA = \text{the included } \angle POB \\ \quad \text{(being rt. } \angle^s) \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 4)

so that $PA=PB$.

Q. E. D.

3. Let ABCD be a square and let AC and BD be its two diagonals.



It is required to prove that the diagonals AC and BD are equal

Proof —In the two $\triangle^s$ ADC and BDC

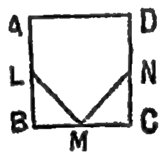
Because
- AD=BC (being sides of a square)
- DC is common to both
- and the incuded $\angle$ ADC=the included $\angle$ BCD (being rt $\angle^s$)

$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 4)

so that AC=BD.

Q E. D.

4 (i) Let ABCD be a square, and let L be the middle point of AB, M the middle point of BC, and N the middle point of CD. Join LM and MN.



It is reqnired to prove that LM and NM are equal.

Proof.—Because AB=CD=BC (being sides of a square)

And AL=LB=½ AB.

and DN=NC=½CD.

$\therefore$ LB=NC

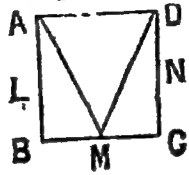
Now, in the two triangles BLM and MNC

Because
- BL=NC (proved)
- BM=MC (by hypothesis.)
- and the included $\angle$LBM=the included $\angle$MCN (being rt. $\angle^s$)

∴ two $\triangle^s$ are equal in all respects. (Theor. 4)

So that LM=MN.

(ii) Let ABCD be a square and let L, M, N be the middle points of AB, BC, CD respectively.



Join MA and MD.

It is required to prove that MA and MD are equal.

Proof.—In the two $\triangle^s$ ABM and DMC

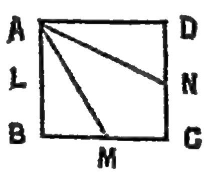
Because $\begin{cases} \text{AB=DC (being sides of a square)} \\ \text{BM=MC (by hypothesis)} \\ \text{and the included } \angle \text{ABM=the included } \angle \text{ DMC} \\ \text{(being rt. } \angle^s\text{)} \end{cases}$

∴ two triangles are equal in all respects. (Theor. 4)

so that AM=DM.

(iii) Let ABCD be a square and let L, M, N, be the middle points of AB, BC, CD respectively.

Join AM and AN.



It is required to prove that AM and AN are equal.

Proof—Because BC=CD (being sides of a square).

And BM=MC=½BC.

and DN=CN=½CD.

∴ BM=DN.

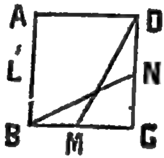
Now, in the two $\triangle^s$ ABM and ADN

Because $\begin{cases} BM = DN \text{ (proved),} \\ AB = AD \text{ (being sides of a square)} \\ \text{and the included } \angle ABM = \text{the included } \angle ADN \\ \text{(being rt. } \angle^s) \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that $AM = AN$

(iv) Let ABCD be a square and let L, M, N be the middle points of AB, BC, CD respectively.



Join BN and DM.

It is required to prove that BN and DM are equal.

Proof—Because $BC = CD$ (being sides of a square)

And $BM = MC = \frac{1}{2}BC$ and $DN = CN = \frac{1}{2}CD$.

$\therefore MC = CN$.

Now in the two $\triangle^s$ BNC and MCD.

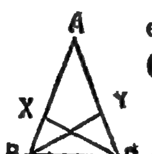
Because $\begin{cases} NC = MC \text{ (proved)} \\ BC = CD \text{ (hypothesis)} \\ \text{and the included } \angle BCN = \text{the included } \angle DCM \\ \text{(being rt. } \angle^s) \end{cases}$

$\therefore$ Two $\triangle^s$ are equal in all respects. (Theor. 4)

so that $BN = DM$

Q. E. D.

5. Let ABC be an isosceles triangle. From the equal sides AB and AC two equal parts AX and AY are cut off. Join BY and CX.



It is required to prove that BY and CX are equal.

Proof.—In the two $\triangle^s$ ABY and ACX

Because { AY = AX (by hypothesis)
AB = AC (being equal sides of isosceles triangle),
and the $\angle$ BAY or $\angle$ CAX is common to both

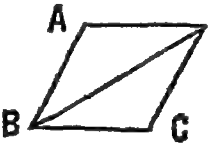
$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 4)

so that, BY = CX.

Q. E. D.

Page 21.

1. Let ABCD be a quadrilateral whose all sides are equal, and let the diagonal BD be joined.



(i) It is required to prove that the $\angle^s$ ABD and ADB are equal.

Proof—Because in the $\triangle$ ABD, AB = AD (by hypothesis)

$\therefore$ the $\angle$ ADB = the $\angle$ ABD. (Theor. 5)

(ii) It is required to prove that the $\angle^s$ CBD and CDB are equal.

Proof—Because in the $\triangle$ CBD, CB = CD (by hypothesis)

$\therefore$ the $\angle$ ABD = the $\angle$ ADB (Theor. 5)

(iii) It is required to prove that the $\angle^s$ ABC and ADC are equal.

Proof—Because in the $\triangle$ ABD, AD = AB (by hypothesis)

$\therefore$ the $\angle$ ADB = the $\angle$ ABD (Theor 5)

Again because in the $\triangle$ CBD CB = CD (by hypothesis)

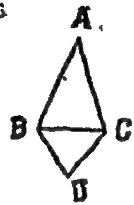
$\therefore$ the $\angle$ ABD = the $\angle$ ADB (Theor. 5)

$\therefore$ the $\angle^s$ ADB and CDB together = the $\angle^s$ ABD and CBD together

Or the $\angle$ ADC = the $\angle$ ABC.

Q. E. D.

2. Let ABC and DBC be two isosceles triangles drawn on the same base BC but on opposite side of it.



It is required to prove that the $\angle^s$ ABD and ACD are equal.

Proof.—Because in the $\triangle$ ABC, AB = AC (being sides of an isosceles triangle)

$\therefore$ the $\angle$ ABC = the $\angle$ ACB (Theor. 5)

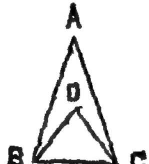
Again because in the $\triangle$ DBC, DB = DC (being sides of an isosceles triangle)

$\therefore$ the $\angle$ DBC = the $\angle$ DCB (Theor. 5)

$\therefore$ the $\angle^s$ ABC and DBC together = the $\angle^s$ ACB and DCB together or the $\angle$ ABD = the $\angle$ ACD

Q. E. D.

3. Let ABC and DBC be two isosceles triangles drawn on the same base BC and on the same side of it.



It is required to prove that the $\angle^s$ ABD and ACD are equal.

Proof.—Because in the $\triangle$ ABC, AB = AC (being sides of an isosceles triangle)

$\therefore$ the $\angle$ ABC = the $\angle$ ACB (Theor. 5)

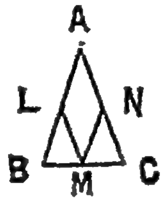
Again because in the $\triangle$ DBC, DB = DC (being sides of an isosceles triangle)

$\therefore$ the $\angle$ DBC = the $\angle$ DCB (Theor. 5)

$\therefore$ the $\angle$ ABC — the $\angle$ DBC = the $\angle$ ACB — the $\angle$ DCB or the $\angle$ ABD = the $\angle$ ACD.

Q. E. D.

4. (*i*) Let ABC be an isosceles triangle of which the sides AB and AC are equal. Let L, M, N be the middle points of AB, BC and CA respectively. Join LM and NM.



It is required to prove that LM and NM are equal.

Proof.—Because AB = AC (by hypothesis),

$\therefore$ the $\angle$ ABC = the $\angle$ ACB (Theor. 5)

AL = LB = $\frac{1}{2}$ AB ; and AN = NC = $\frac{1}{2}$ AC

$\therefore$ LB = NC.

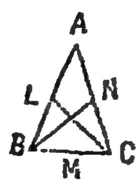
Now, in the two $\triangle$s LBM and MCN

Because $\begin{cases} \text{LB = NC (proved)} \\ \text{BM = MC (by hypothesis)} \\ \text{and the included } \angle \text{ LBM = the included } \angle \text{ NCM} \end{cases}$

$\therefore$ two $\triangle$s are equal in all respects. (Theor. 4)

so that LM = MN.

(*ii*) Let ABC be an isosceles triangle whose sides AB and AC are equal and let L, M, N be the middle points of AB, BC and CA respectively. Join BN and CL.



It is required to prove that BN and CL are equal.

Proof —Because in the $\triangle$ ABC, AB=AC (being sides of an isosceles triangle)

$\therefore$ the $\angle$ ABC=the $\angle$ ACB (Theor. 5)

AL=LB =$\frac{1}{2}$AB, and AN=NC=$\frac{1}{2}$ AC

$\therefore$ AL=AN Now, in the two $\triangle^s$ ABN and ALC

Because $\begin{cases} \text{AN=AL (proved)} \\ \text{AB=AC (being sides of an isosceles triangle)} \\ \text{and the included } \angle \text{ BAN or CAL is common} \\ \text{to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor 4)

so that BN=CL.

(*iii*) (See Fig in Ex. 4 (*i*).

Let ABC be an isosceles triangle whose sides AB and AC are equal and let L, M, N, be the middle points of AB, BC and CA respectively. Join LM and NM.

It is required to prove that the $\angle^s$ ALM and ANM are equal

Proof —Because in the $\triangle$ ABC, AB=AC (being sides of an isosceles triangle)

$\therefore$ the $\angle$ ABC=the $\angle$ ACB. (Theor. 5)

But AL=LB=$\frac{1}{2}$ AB, and AN=NC=$\frac{1}{2}$ AC

$\therefore$ LB = NC.

Now, in the $\triangle^s$ LBN and NMC

Because $\begin{cases} \text{LB=NC (proved)} \\ \text{BM=MC (by hypothesis)} \\ \text{and the included } \angle \text{ LBM=the included } \angle \text{ NCM} \\ \text{proved} \end{cases}$

$\therefore$ two $\angle^s$ are equal in all respects (Theor., 4)

so that the $\angle$ BLM=the $\angle$ MNC.

Because LM stands on AB at L

∴ the $\angle^s$ ALM and MLB together $=$ 2 rt. $\angle^s$ (Theor. 1)

Also, because NM stands on AC at N

∴ the $\angle^s$ ANM and MNC together $=$ 2 rt. $\angle^s$ (Theor. 1)

∴ the $\angle^s$ ALM and MLB $=$ the $\angle^s$ ANM and MNC.

But the $\angle$ MLB $=$ the $\angle$ MNC (proved)

∴ the $\angle$ ALM $=$ the $\angle$ ANM.

Q E. D.

Page 26.

1. (See Fig. in Ex. 4, p. 19.)

Let ABC be an isosceles triangle and let D be the middle point of BC. Join AD.

(i) It is required to prove that AD bisects the vertical $\angle$ BAC.

Proof.—In the two triangles ABD and ACD

Because $\begin{cases} \text{AB} = \text{AC (being sides of an isosceles triangle)} \\ \text{BD} = \text{DC (by hypothesis)} \\ \text{and AD is common to both} \end{cases}$

∴ two $\triangle^s$ are equal in all respects. (Theor. 7)

so that the $\angle$ BAD $=$ the $\angle$ DAC

i. e. the vertical $\angle$ BAC is bisected by AD.

(ii) It is required to prove that AD is perpendicular to the base BC.

Proof —In the two $\triangle^s$ ABD and ACD.

Because $\begin{cases} \text{AB} = \text{AC (being sides of an isosceles triangle)} \\ \text{BD} = \text{DC (by hypothesis)} \\ \text{and AD is common to both} \end{cases}$

∴ two $\triangle^s$ are equal in all respects. (Theor. 7)

so that, the ∠ ADB = the ∠ ADC and these being adjacent angles, each is rt. ∠.

∴ AD is perpendicular to BC.

Q. E. D.

2. Let ABCD be a rhombus and let the diagonal AC be joined.

A D

B C

(i) It required to prove that the ∠^{s} ABC and ADC are equal.

Proof—In the two △^{s} ABC and ADC

Because $\begin{cases} \text{AB}=\text{AD (being sides of a rhombus)} \\ \text{BC}=\text{CD (being sides of a rhombus) and AC is} \\ \quad \text{common to both} \end{cases}$

∴ two △^{s} are equal in all respects (Theor. 7)

so that the ∠ ABC = the ∠ ADC.

(ii) It is required to prove that the angles BAD and BCD are bisected by AC.

Proof.—In the two △^{s} ABC and ADC

Because $\begin{cases} \text{AB}=\text{AD (being sides of a rhombus)} \\ \text{BC}=\text{CD (being sides of a rhombus)} \\ \text{and AC is common to both} \end{cases}$

∴ two △^{s} are equal in all respects. (Theor. 7)

so that the ∠ DAC = the ∠ BAC and the ∠ ACD = the ∠ ACB i. e. the ∠^{s} DAB and DCB are bisected by AC.

Q. E. D.

3. Let ABCD be a quadrilateral in which AB = CD and AD = CB Join AC.

A

D

B

C

It is required to prove that the $\angle^s$ ADC and ABC are equal.

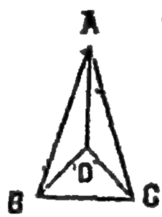
Proof.—In the two $\triangle^s$ ADC and ABC

Because $\begin{cases} AD=BC \text{ (by hypothesis)} \\ DC=AB \text{ (by hypothesis)} \\ \text{and } AC \text{ is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 7)
so that the $\angle$ ADC=the $\angle$ ABC.

Q. E. D.

4 (*i*) Let ABC and DBC be two isosceles triangles standing on the same base BC and on the same side of it.



It is required to prove that the $\angle^s$ ABD and ACD are equal.

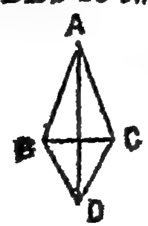
Join AD

Proof.—In the two $\triangle^s$ BAD and CAD,

Because $\begin{cases} BA=AC \text{ (being sides of isosceles } \triangle \text{ ABC)} \\ BD=DC \text{ (being sides of isosceles } \triangle \text{DBC)} \\ \text{and AD is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 7)
so that the $\angle$ ABD=the $\angle$ ACD.

(*ii*) Let ABC and DBC be two isosceles triangles standing on the same base BC but on opposite sides of it.



It is required to prove that the $\angle^s$ ABD and ACD are equal

Join AD

Proof —In the tws $\triangle^s$ BAD and CAD

Because $\begin{cases} AB=AC \text{ (being sides of an isosceles triangle)} \\ BD=DC \text{ (being sides of an isosceles triangle)} \\ \text{and } AD \text{ is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 7)

so that the $\angle$ ABD=the $\angle$ ACD.

Q. E. D.

5. (See. fig. in Ex 4 (*i*)).

Let ABC and DBC be two isosceles triangles standing on the same base BC but on opposite sides of it and let AD be joined.

It is required to prove that the $\angle^s$ BAC and BDC are bisected by AD

Proof —In the two $\triangle^s$ BAD and DAC

Because $\begin{cases} BA=AC \text{ (being sides of an isosceles triangle)} \\ BD=DC \text{ (being sides of an isosceles triangle)} \\ \text{and } AD \text{ is common to both} \end{cases}$

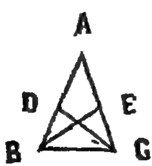
$\therefore$ two $\triangle^s$ are equal in all respects (Theor 7)

so that the $\angle$ BAD=the $\angle$ CAD and the $\angle$ BDA=the $\angle$ CDA

i. e. the $\angle^s$ BAC and BDC are bisected by AD.

Q. E. D.

6. Let ABC be an isosceles triangle and let D and E be the middle points of AB and AC respectively. Let BE and CD be joined.



It is required to prove that BE and CD are equal.

Proof.— In the $\triangle$ ABC, because AB=AC (being sides of an isosceles triangle)

$\therefore$ the $\angle$ ABC=the $\angle$ ACB (Theor. 5)

AD=DB=$\frac{1}{2}$ AB, and AE=EC=$\frac{1}{2}$ AC

$\therefore$ DB=EC.

Now, in the two $\triangle^s$ DBC and EBC

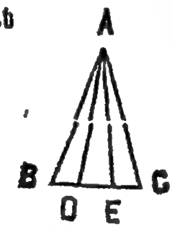
Because $\begin{cases} \text{DB=EC (proved)} \\ \text{BC is common to both} \\ \text{and the included } \angle \text{ DBC=the included } \angle \text{ ECB} \\ \text{(proved)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 4)

so that the side DC=the side BE.

Q. E. D.

7. Let ABC be an isosceles triangle, and let D, E be two such points in BC that BD=EC. Join AD and AE.



It is required to prove that AD=AE

Proof—Because in the $\triangle$ ABC, AB=AC (being sides of an isosceles triangle)

$\therefore$ the $\angle$ ABC=the $\angle$ ACB (Theor. 5)

Now, in the two $\triangle^s$ ABD and AEC

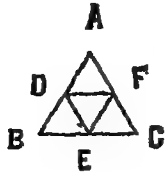
Because $\begin{cases} \text{AB=AC (being sides of in isosceles triangle)} \\ \text{BD=EC (by hypothesis)} \\ \text{and the included } \angle \text{ ABD=the included } \angle \text{ ACE} \\ \text{(proved)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 4)

so that AD=AE.

Q. E. D.

8. Let ABC be an equilateral triangle and let D, E F be the middle points of AB, BC and CA respectively. Join DE, EF and DF.



It is required to prove that DEF is an equilateral triangle.

Proof —Because the △ ABC is equilateral,

∴ the ∠ ACB=the ∠ BAC=the ∠ ABC
(Theor. 5, Cor. 2)

In the △ ABC, AB=AC=BC (being sides of an equilateral triangle)

AD=DB=½ AB; BE=EC=½ BC; and AF=FC=½ AC

∴ AD=DB=BE=EC=AF=FC

Now, in the two △s ADF and BDE

Because { AD=BD (by hypothesis)
AF=BE (proved)
and the ∠ =DBEthe ∠ DAF (proved)

∴ two △s are equal in all respects. (Theor. 4)

so that DF=DE

Again in the two △s ADF and FEC

Because { AF=FC (by hypothesis)
AD=EC (proved)
and the ∠ DAF=the ∠ FCE (proved)

∴ two △s are equal in all respects. (Theor. 4)

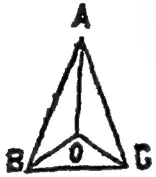
so that DF=FE

$\therefore$ DE=FE=DF

Hence, DEF is an equilateral triangle.

Q. E. D.

9. Let ABC be an isosceles triangle and let the angles ABC and ACB be bisected by BO and CO respectively.



(*i*) It is required to prove that BO and CO are equal.

Join AO

Proof—In the $\triangle$ ABC, because AB=AC

$\therefore$ the $\angle$ ABC=the $\angle$ ACB (Theor. 5)

the $\angle$ OBC=$\frac{1}{2}$ the $\angle$ ABC (given)

and the $\angle$ OCB=$\frac{1}{2}$ the $\angle$ ACB (given)

$\therefore$ the $\angle$ OBC=the $\angle$ OCB

$\therefore$ OB=OC (Theor. 6)

(*ii*) It is required to prove that the $\angle$ BAC is bisected by AO.

Join AO.

Proof.—in the $\triangle$ ABC, because AB=AC

$\therefore$ the $\angle$ ABC=the $\angle$ ACB (Theor. 5)

the $\angle$ OBC=$\frac{1}{2}$ the $\angle$ ABC (given)

and the $\angle$ OCB=$\frac{1}{2}$ the $\angle$ ACB (given)

$\therefore$ the $\angle$ OBC= the $\angle$ OCB

$\therefore$ OB=OC (Theor. 6)

Now, in the two $\triangle^s$ AOB and AOC

Because $\begin{cases} \text{AB=AC (given)} \\ \text{BO=OC (proved)} \\ \text{and AO is common to both} \end{cases}$

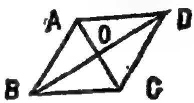
∴ two $\triangle^s$ are equal in all respects. (Theor. 7)

so that the $\angle$ BAO = the $\angle$ CAO.

i. e. AO bisects the $\angle$ BAC.

Q. E. D.

10 Let ABCD be a rhombus and let the diagonals AC and BD cut at O.



It is required to prove that the diagonals AC and BD bisect one another at right angles at O

Proof —In the two $\triangle^s$ ABC and ADC

Because $\begin{cases} AB = AD \text{ (by hypothesis)} \\ BC = CD \text{ (by hypothesis)} \\ \text{and } AC \text{ is common to both} \end{cases}$

∴ two $\triangle^s$ are equal in all respects. (Theor. 7)

so that the $\angle$ BCA = the $\angle$ DCA.

Again, in the two $\triangle^s$ BOC and DOC.

Because $\begin{cases} BC = CD \text{ (by hypothesis)} \\ CO \text{ is common to both} \\ \text{and the } \angle BCO = \text{the } \angle DCO \text{ (proved)} \end{cases}$

∴ two $\triangle^s$ are equal in all respects (Theor. 4)

so that BO = OD and the $\angle$ BOC = the $\angle$ COD, and these being adjacent angles each is a rt $\angle$.

Again, in the two $\triangle^s$ ADB and DCB.

Because $\begin{cases} AD = DC \text{ (by hypothesis)} \\ AB = BC \text{ (by hypothesis)} \\ \text{and } BD \text{ is common to both} \end{cases}$

∴ two $\triangle^s$ are equal in all respects. (Theor. 7)

so that the $\angle$ ADB = the $\angle$ CDB

Again, in the two $\triangle^s$ AOD and DOC.

Because $\begin{cases} AD = DC \text{ (by hypothesis)} \\ DO \text{ is common to both} \\ \text{and the } \angle\, ADO = \text{the } \angle\, CDO \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 4)

so that $AO = OC$ and the $\angle\, AOD =$ the $\angle\, DOC$, and these being adjacent angles each is a rt. $\angle$.

Because AC and BD cut one another at O.

$\therefore$ The $\angle\, AOB =$ the $\angle\, DOC$ (Theor. 3)

But the $\angle\, DOC$ is a rt. $\angle$ (proved)

$\therefore$ The $\angle\, AOB$ is a rt. $\angle$

i.e., the $\angle^s$ BOA, AOD, DOC and COB are each a rt. $\angle$

and $BO = OD$, $AO = OC$

$\therefore$ the diagonals AC and BD bisect one another at right angles at O.

Q. E. D.

11. Let ABC be an isosceles triangle and let the equal sides BA, CA be produced to any points E and F beyond the vertex A, such that AE is equal to AF.

Let FB and EC be joined.

It is required to prove that FB and EC are equal.

Proof —In the two $\triangle^s$ ABF and ACE,

Because $\begin{cases} AB = AC \text{ (given)} \\ AF = AE \text{ (given)} \\ \text{and the } \angle\, BAF = \text{the } \angle\, CAE \quad \text{(Theor. 3)} \end{cases}$

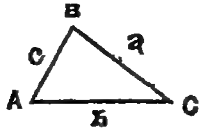
$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 4)

so that $FB = EC$.

Q. E. D.

Page 27.

1. Draw a straight line AC=2·1″. With C and A as centres and the radii equal to 2 0″ and 1·3″ draw two arcs cutting at B.



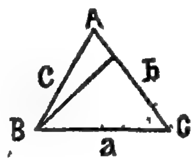
Join BA and BC.

Then ABC is the required triangle.

Measure the $\angle^s$ ABC, ACB and BAC, and see that the $\angle$ BAC=68°, the $\angle$ ACB=37° and the $\angle$ ABC=75°.

The sum of the $\angle^s$ ABC, ACB and BAC=68°+37°+75°.
=180°.

2 Draw a straight line AC=7 cm. With A and C, as centres and the radii equal to 6·5 cm, and 7·5 cm. draw two arcs cutting at B.



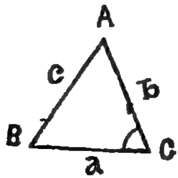
Join BA and CB.

Then ABC is the required triangle.

From B drop a perpendicular BD to CA.

Measure BD and it will be found to be equal to 6 cm

3 Construct an angle BCA=65° of which the arm BC is equal to 7 cm. and AC equal to 6 cm.



Join AB.

Then ABC is the required triangle.

Theoretically any two triangles having these parts would have two sides of the one equal to two sides of the other as well as their included angles equal.

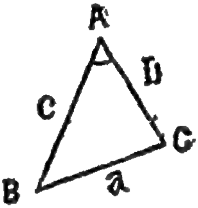
Then the triangles would be equal in all respects.

(Theor. 4)

∴ The triangles would be alike in size and shape.

The above statement can be experimentally illustrated by cutting two such triangles from a piece of paper and by superposing one of the triangles on the other when they will exactly coincide.

4. Make an angle ABC=57° whose arm BA=2.5″ and AC=2″.



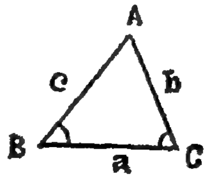
Join BC.

Then ABC is the required triangle.

Measure the side BC, the ∠ ABC and the ∠ ABC.

It will be found that BC=2·2″, the ∠ ABC=50° and the ∠ ACB=73° nearly.

Draw a straight line BC=2·2″. At B and C make angles ABC and ACB= 50° and 73° respectively cutting at A.



Then ABC is the required triangle.

Measure the sides BA and AC and the ∠ BAC

It will be found that BA=2 5″, AC=2″ and the ∠ BAC =57° (very nearly)

∴ the triangles constructed in both cases are indentically equal.

5. Draw a veritical line AB=3·5″ representing the height of the window above the ground From B draw BC=1 2″ perpendicular to AB which is the distance of the ladder from the base of the house.

Join AC which represents the ladder.

Measure AC and it will be found to be equal to 3.7″.

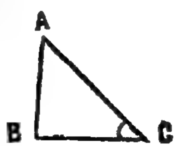
∴ The ladder is 37 ft. long.

6. Let A be any point representing the starting point. From A draw AB=9 9 cm. vertically upwards. From B draw BC=2 cm perpendicular to AB. Then C represents the final position. Join CA and measure it.

It will be found to be equal to 10 1 cm.

∴ The distance from the starting point is 101 metres.

7. Draw a horizontal line BC=3″ At C make an angle BCA=42°. From B draw BA perpendicural to BC meeting CA in A.



Then BC represents the shadow, AB the direction of the rays of sun and AB the vertical pole

Measure BA and it will be found 2·7″ long

∴ the pole is 27 ft. long

8. Let A be the starting point of the surveyor. From A draw AB=1·5″ to the right. At B drop a perpendicular BC=3″ to AB vertically upwards. From C draw CD=4·5″ perpendicular to BC to the left. Join DA.

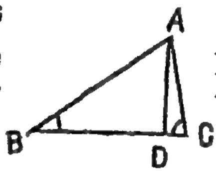
Measure DA and it will be found to be 4·24″.

∴ the distance of the point D from the starting point A is 424 yards.

Also measure the ∠ DAB and it will be found to be 135°.

∴ the point D bears a north westerly direction from the point A.

9 Let B and C be two points at a distance of 2·6 cm apart. Join BC and at the point B make an ∠ CBA=33° and at the point C an ∠ BCA=81°, the two arms of the angles CBA and BCA meeting at A.



Then A represents the position of the vessel.

Measure AB and AC. It will be found that AB=2·81″ and AC=1·55″.

∴ the vessel is 281 yards from B and 155 yards from C.

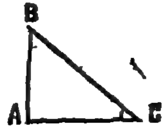
Drop AD perpendicular to BC.

Then D represents the nearest point on the shore.

Measure AD and it will be found 1·53″ long.

∴ the vessel is 153 yard from the nearest point on the shore.

10 Make an angle ACB=24° having the sides AC, CB=2·45″ and 3·2″ respectively.



Then A and B represent the two points in the park between which the lake intervenes, and C represents the third point from which both A and B are accessible.

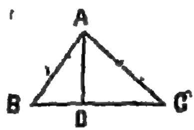
Join AB and measure it.

It will be found that AB=2 14″

∴ the distance between the points A and B is 214 yards.

Page 29.

1. Let ABC be any triangle.



It is required to prove that any two angles of the △ ABC are together less than two right angles.

Take any point D in BC Join AD.

Proof—Because in the △ ABD, BD is produced to C

∴ the exterior ∠ ADC is greater than the interior opposite ∠ ABD, or ABC (Theor 8)

Again, because in the △ ADC, DC is produced to B

∴ the exterior ∠ ADB is greater than the interior opposite ∠ ACD, or ACB (Theor. 8)

∴ the ∠s ABC and ACB are together less than the ∠s ADC and ADB.

But the $\angle^s$ ADC and ADB together = 2 rt. $\angle^s$.
(Theor. 1)

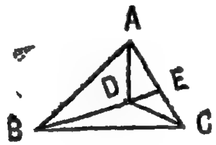
$\therefore$ the $\angle^s$ ABC and ACB of the $\triangle$ ABC are together less than 2 rt $\angle^s$.

Q. E. D.

2. Let ABC be a triangle and D be any point within it. Join BD and CD.

It is required to prove that the $\angle$ BDC is greater than the $\angle$ BAC.

(*i*) Produce BD beyond D to meet AC in E.



In the $\triangle$ ABE, AE is produced to C

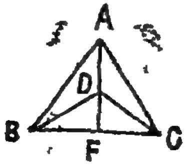
$\therefore$ the exterior $\angle$ BEC, or $\angle$ DEC is greater than the interior opposite $\angle$ BAE, or $\angle$ BAC (Theor. 8.)

Again, in the $\triangle$ DEC, ED is produced to B

$\therefore$ the exterior $\angle$ BDC is greater than the interior opposite $\angle$ DEC.

$\therefore$ still more is the $\angle$ BDC greater than the $\angle$ BAC.

(*ii*) Join AD and produce AD beyond D to meet BC in F.



In the $\triangle$ ABD, AD is produced to F

$\therefore$ the ext. $\angle$ BDF is greater than the int. opp. $\angle$ BAD (Theor. 8)

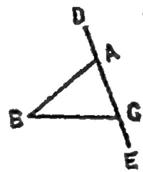
In the $\triangle$ ADC, AD is produced to F

$\therefore$ the ext. $\angle$ FDC is greater than the int. opp. $\angle$ CAD (Theor. 8)

$\therefore$ the whole $\angle$ BDC is greater than the whole $\angle$ BAC.

Q. E. D.

3. Let ABC be any triangle and let the side AC be produced both ways to the points D and E.



It is required to prove that the exterior angles BAD and BCE so formed are together greater than 2 rt. $\angle^s$.

The $\angle^s$ BAD and BAC together = 2 rt $\angle^s$ (Theor. 1)

and the $\angle^s$ BCE and BCA together = 2 rt $\angle^s$ (Theor 1)

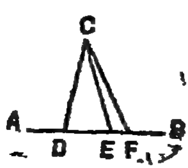
$\therefore$ the $\angle^s$ BAD, BAC, BCE and BCA together = 4 rt $\angle^s$.

But in the $\triangle$ ABC, the $\angle^s$ BAC and BCA are together less than 2 rt. $\angle^s$ (Cor. 1. Theor 8)

$\therefore$ the $\angle^s$ BAD and BCE are together greater than 2 rt $\angle^s$.

Q. E. D.

4 Let AB be a given straight line and let C be a given point outside it.



It is required to prove that there cannot be drawn more than two straight lines of the same given length from C to AB.

Draw two equal straight lines CD and CE to AB and if possible let CF be another straight line equal to CD or CE drawn from C to AB.

Proof.—Because CD=CE (by construction)

∴ the ∠ CDE=the ∠ CED (Theor. 5)

Again, because CD=CF (by supposition)

∴ the ∠ CDF=the ∠ CFD (Theor. 5)

∴ the ∠ CED=the ∠ CFD

i. e, the ext. ∠ CED=the int. opp. ∠ CFD which is absurd according to Theor 8

∴ CF is not equal to CE or CD.

∴ CE and CD are the only two equal straight lines drawn from C to AB.

Q. E. D.

5. See Fig. in Ex. 5 on p 13.

Let ABC be an isosceles triangle and let the equal sides AB and AC be produced to any points D and E respectively.

It is required to prove that the exterior ∠s CBD and BCE thus formed are each an obtuse angle.

Proof —Because in the △ ABC, AB=AC (given)

∴ the ∠ ABC=the ∠ ACB (Theor. 5)

But the ∠s ABC and ACB are together less than 2 rt. ∠s

(Cor. 1. Theor. 8)

Hence, each of the ∠s ABC and ACB is acute

But the ∠s ABC and CBD together=2 rt. ∠s (Theor. 1)

the ∠ ABC is acute, hence the ∠ CBD is obtuse.

Also the ∠s ACB and BCE together=2 rt. ∠s.

(Theor. 1)

but the $\angle$ ACB is acute, hence the $\angle$ BCE is obtuse.

i.e, each of the ext. $\angle^s$ CBD and BCE is obtuse.

Q E D.

Page 34.

1. Let ABC be a right-angled triangle, right angled at A.

It is required to prove that the hypotenuse BC is the greatest side.

Proof.—In the $\triangle$ ABC, the $\angle$ BAC is a rt $\angle$

hence, each of the $\angle^s$ ABC and ACB is acute

(Cor 2 Theor. 8)

$\therefore$ the $\angle$ BAC is the greatest angle.

Because the $\angle$ BAC is greater than the $\angle$ ABC

$\therefore$ the side BC is greater than the side AC (Theor 10)

Again, because the $\angle$ BAC is greater than the $\angle$ ABC

$\therefore$ the side BC is greater than the side AB (Theor. 10)

$\therefore$ BC is the greatest side.

Q. E. D.

2. Let ABC be a triangle in which BC is the greatest side.

It is required to prove that the side BC makes acute angles with each of the other sides AB and AC, i.e., the angles ABC and ACB are acute.

Proof—Because BC is greater than AB given)

$\therefore$ the $\angle$ BAC is greater than the $\angle$ ACB (Theor. 9)

But the $\angle^s$ BAC and ACB are together less than two right angles (Cor. Theor. 8)

∴ the ∠ ABC is less than a right angle, or, is an acute angle.

Similarly, it can be proved that the ∠ ABC is an acute angle.

Q. E. D.

3. See Fig. in Ex. 2 (*i*) on page 29.

Let ABC be a triangle and let two straight lines BD and CD be drawn from the ends B,C, to meet at D within the triangle.

It is required to prove that BD and DC are together less than BA and AC.

Produce BD to meet AC in E.

Proof.—In the △ABE, AB and AE *are together greater* than BE (Theor 11)

By adding EC to both, we have

AB and (AE+EC) *i.e.*, AB and AC together greater than BE and EC.

Again, in the △ DEC, DE and EC are together greater than DC (Theor. 11)

By adding BD to both, we have

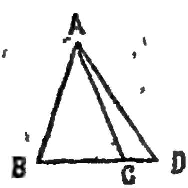
BE (*i e*, BD+DE) and EC together greater than BD and DC

∴ Still more are AB and AC greater than BD and DC

or, BD and DC are together less than AB and AC.

Q. E. D.

4. Let ABC be an isosceles triangle of which the base BC is produced to any point D and let AD be joined.



It is required to prove that AD is greater than either of the equal sides AB and AC of the △ ABC.

Proof.—In the △ ACD, DC is produced to B

∴ the ext. ∠ ACB is greater than the int. opp. ∠ADB

∴ the ∠ABC is greater than the ∠ ADB (∵ ∠ACB= ∠ABC by Theor. 5)

(Theor. 8)

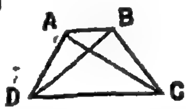
∴ the side AD is greater than the side AB (Theor. 10)

but AB=AC

∴ the side AD is also greater than the side AC.

Q E. D.

5. Let ABCD be a quadrilateral in which the greatest side DC and the least side AB are opposite to one another.



It is required to prove that the ∠ABC is greater than the ∠ADC and the ∠DAB is greater than the ∠BCD.

Join BD

Proof.—In the △ ADB, AD is greater than AB (given)

∴ the ∠ ABD is greater than the ∠ ADB (Theor. 9)

Again, in the △ BDC, DC is greater than BC (given)

∴ the ∠ BDC is greater than the ∠ BDC (Theor. 9)

∴ the ∠s ABD and DBC are greater than the ∠s ADB and BDC

or, the ∠ ABC is greater than the ∠ ADC.

Similarly, by joining AC, it can be proved that the ∠ DAB is greater than the ∠ BCD.

Q. E. D.

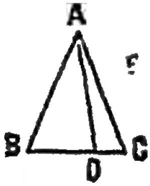
6. Let ABC be a triangle in which AC is not greater than AB, and let AD be any straight line drawn through the vertex A and terminated by the base BC.

It is required to prove that AD is less than AB.

If AC is not greater than AB, it must be either

(i) equal to or (ii) less than AB.

(i) If AC is equal to AB.



then the $\angle$ ACB = the $\angle$ ABC (Theor. 5)

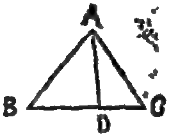
But the ext. $\angle$ ADB is greater than the int. opp. $\angle$ ACB (Theor. 8.)

$\therefore$ the $\angle$ ADB is also greater than the $\angle$ ABC

$\therefore$ the side AB is greater than the side AD (Theor. 10)

or, AD is less than AB.

(ii) If AC were less than AB



then the $\angle$ ABC is less than the $\angle$ ACB (Theor. 9)

But the ext. $\angle$ ADB is greater than the int. opp. $\angle$ ACB (Theor. 8)

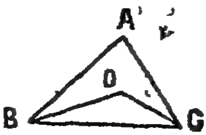
$\therefore$ the $\angle$ ADB is much greater than the $\angle$ ABC

$\therefore$ AB is greater than AD (Theor. 10)

or, AD is less than AB.

Q. E. D.

7. Let ABC be a triangle in which the side AB is greater than the side AC. Let the angles ABC and ACB be bisected by the lines BO and CO meeting at O.



It is required to prove that OB is greater than OC

Proof —Because AB is greater than AC (given)

∴ the ∠ ACB is greater than the ∠ ABC (Theor. 9)

But the ∠ OBC $=\frac{1}{2}$ of the ∠ ABC, and the ∠ OCB $=\frac{1}{2}$ of the ∠ ACB (given)

∴ the ∠ OCB is greater than the ∠ OBC

∴ OB is greater than OC (Theor. 10).

Q. E D.

8. (See Fig in Ex. 2 on p. 34)

Let ABC be a triangle in which the side BC is the greatest side

It is required to prove that the difference of any two sides of the △ ABC is less than the third side.

Proof —In the △ ABC, BA and AC are together greater than BC (Theor. 11)

Subtracting AC from both we get

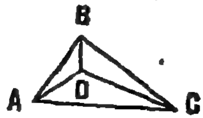
AB greater than BC − AG

i. e, AB is greater than the difference of AC and BC.

Similarly, it can be proved that AC is greater than the difference of BC and AB, and BC being the greatest side it is evidently greater than the difference of AB and AC.

Q. E D.

9 Let ABC be a triangle and let D be any point within the triangle, Join BD, AD and CD.



It is required to prove that

(DB + DA + DC) is greater than $\frac{1}{2}$ (AB + BC + CA)

Proof—In the △ ABD, (AD + DB) is greater than AB (Theor. 11).

Similarly, in the $\triangle$ ACD

(AD+CD) is greater than AC, and in the $\triangle$ BCD,

$\angle$ (BD+CD) is greater than BC.

$\therefore$ by summing up these three results, we get

2 (DB+DA+DC) greater than (AB+BC+CA)

$\therefore$ (DB+DA+DC) is greater than $\frac{1}{2}$ (AB+BC+CA)

Q. E. D.

10 (see fig. in Ex. 5 on p. 34).

Let ABCD be a quadrilateral and let AC and BD be its diagonals.

It is required to prove that

(AB+BC+CD+DA) is greater than (AC+DB).

Proof.—In the $\triangle$ ABD, (DA+AB) is greater than DB (Theor. 11)

Similarly, in the $\triangle$ DCB, (BC+CD) is greater than DB

in the $\triangle$ ADC, (AD+DC) is greater than AC,

and in the $\triangle$ ACB, (AB+BC) is greater than AC.

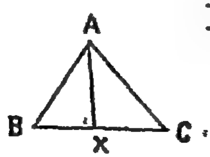
$\therefore$ summing up these four results, we get

2 (AB+BC+CD+DA) greater than 2 (AC+BD)

$\therefore$ (AB+BC+CD+DA) is greater than (AC+BD)

Q. E. D.

11. Let ABC be a triangle and let AX bisect the vertical angle BAC meeting BC in X.



It is required to prove that BA is greater than BX and CA greater than CX.

Proof —In the $\triangle$ ABX, BX is produced to C

$\therefore$ the ext. $\angle$ AXC is greater than the int. opp. $\angle$ BAX

But the $\angle$ BAX = the $\angle$ CAX (given)

$\therefore$ the $\angle$ AXC is greater than the $\angle$ CAX

$\therefore$ AC is greater than CX (Theor. 10)

Similarly, it can be proved that BA is greater than BX

By adding these two last results, we get

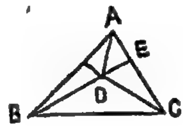
AB and CA greater than BX and CX

i. e., AB and CA are together greater than BC.

Thus we obtain another proof of Theorem 11.

Q. E. D.

12 Let ABC be a triangle and let D be any point within the triangle. Join DA, DB and DC.



It is required to prove that (DA + DB + DC) is less than (AB + BC + CA).

Produce BD to meet AC in E.

Proof.—In the $\triangle$ ABE, AB and AE are together greater than BE (Theor. 11)

Adding EC to both, we have

AB and AC (i e, EA + EC) greater than BE and EC.

In the $\triangle$ DEC, DE and EC are together greater than DC (Theor. 11)

Adding BD to both, we have

BE (i e., DE + BD) and EC greater than DC and BD

$\therefore$ AB and AC are much greater than DC and BD.

Similarly, by producing CD to meet BA it can be proved that AB and BC are greater than DA and DC, and that AC and BC are greater than AD and BD.

Summing up these three results, we have

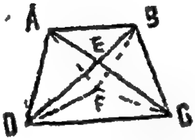
2 (AB+BC+AC) greater than 2 (DA+DB+DC)

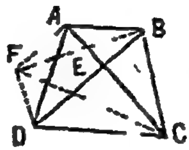
or, (AB+BC+AC) greater than (DA+DB+DC)

i. e., (DA+DB+DC) is less than (AB+BC+AC)

Q E D.

13. Let ABCD be a quadrilateral whose diagonals AC and BD cut one another at E. Let F be any given point within the quadrilateral as in the 1st. figure or, outside the quadrilateral as in the 2nd. figure





It is required to prove that (AC+BD) is less than (FA+FB+FC+FD).

Proof.—In the △ AFC, FA and FC are together greater than AC (Theor 11)

In the △ DFB, FD and FB are together greater than BD (Theor. 11)

Summing up these two results we get

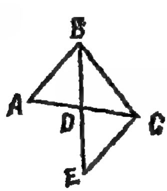
(FD+FB+FA+FC greater than (AC+BD)

or, (AC+BD) is less than (FA+FB+FC+FD)

When the point F coincides with E the point of intersection of the diagonals AC and BD, the proposition fails.

Q. E. D.

14. Let ABC be a triangle and let BD be the median to AC.



It is required to prove that AB and BC are together greater than twice the median BD

Produce BD to any point E making DE=DB. Join CE.

Proof —In the two $\triangle^s$ ADB and DEC

Because $\begin{cases} AD=DC \text{ (given)} \\ BD=DE \text{ (by construction)} \\ \text{and the } \angle ADB = \text{the } \angle EDC \text{ (Theor. 3)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor 4)

so that, AB=EC.

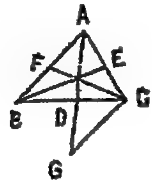
In the $\triangle$ BEC, BC and CE are together greater than BE (Theor. 11)

or, BC and CE are greater than 2 BD

$\therefore$ BC and AB are greater than 2 BD

Q. E. D.

15. Let ABC be a triangle and let AD, BE, CF be its medians.



It is required to prove that (AD+BE+CF) is less than (AB+BC+CA).

Produce AD to any point G making DG=AD.

Join CG.

Proof.—In the $\triangle^s$ ABD and DGC

Because $\begin{cases} BD=DC \text{ (given)} \\ AD=DG \text{ (by construction)} \\ \text{and the } \angle ADB = \text{the } \angle GDC \text{ (Theor. 3)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, AB=GC.

In the $\triangle$ AGC, AC and CG are together greater than AG (Theor. 11)

or, AC and CG are together greater than 2 AD

$\therefore$ AC and AB are greater than 2 AD

Similarly, BA and BC are greater than 2 BE

and, CA and CB are greater than 2 CF

Summing up these three results, we have

2 (AB+BC+CA) greater than 2 (AD+BE+CF)

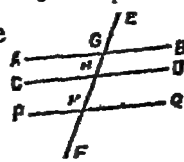
$\therefore$ (AB+BC+CA) is greater than (AD+BE+CF)

or, (AD+BE+CF) is less than (AB+BC+CA)

Q. E. D.

Page 41.

1. Let AB, CD and PQ be parallel to one another and let EF be a straight line cutting AB, CD, PQ at G, H, K respectively.



If the $\angle$ EGB be 55°

It is required to find each of the $\angle^s$ GHC, HKQ, QKF in degrees.

Because AB and CD are parallel and EF cuts them, therefore the ext. $\angle$ EGB = the int. opp. $\angle$ GHD on the same side of the line EF (Theor. 14)

$\therefore$ the $\angle$ GHD = 55°

The $\angle$ GHC is supplement of the $\angle$ GHD

$\therefore$ the $\angle$ GHC = 180° − 55° = 125°.

Because AB and PQ are parallel and EF cuts them

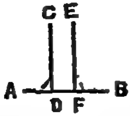
$\therefore$ the EGB = the $\angle$ GKQ or HKQ (Theor. 14)

But the $\angle$ EGB = 55°, $\therefore$ the $\angle$ HKQ = 55°.

The $\angle$ HKQ is supplement of the $\angle$ QKF

But the $\angle$ HKQ = 55°, $\therefore$ the $\angle$ QKF = 180° − 55° = 125°.

2. Let AB be a straight line and let DC, EF be any two straight lines perpendicular to AB.



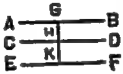
It is required to prove that CD and EF are parallel.

Proof —Because AB cuts two straight lines CD and EF and it makes the interior angle CDF equal to the exterior angle EFB

∴ CD and EF are parallel (Theor. 13)

Q. E. D.

3 Let the straight line GK cut three parallel straight lines AB, CD and EF at the points G, H and K respectively and let it be perpendicular to AB at G.



It is required to prove that GK is also perpendicular to ED and to EF.

Proof.—Because AB and DC are parallel and GK cuts them

∴ the ∠AGH=the alternate ∠ GHD (Theor. 14)

But the ∠ AGH=a rt. ∠

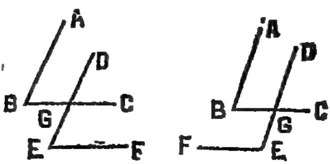
∴ the ∠ GHD is a rt. ∠

i. e., GH is perpendicular to CD.

similarly, it can be proved that GK is perpendicular to EF and to any straight line parallel to EF.

Q. E. D.

4. Let ABC and DEF be any two angles whose arms are parallel, each to each, that is, AB is parallel to DE and BC parallel to EF.



It is required to prove that the angles ABC and DEF are either equal or supplementary.

Let BC and DE cut at G.

Proof.—In the first figure, because AB and GD are parallel and BC meets them

∴ the ext ∠ DGC = the int. opp. ∠ ABC (Theor. 13)

Again, because GC and EF are parallel and DE meets them

∴ the ext ∠ DGC = the int opp. ∠ DEF (Theor. 13)

∴ the ∠ ABC = the ∠ DEF

In the second figure, because AB and DE are parallel and BC cuts them

∴ the ∠ ABC = the alternate ∠ BGE (Theor. 13)

Again, because BG and EF are parallel and DE meets them

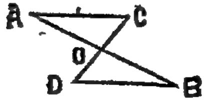
∴ the two int. ∠s BGE and DEF are together equal to 2 rt. ∠s

∴ the ∠s ABC and DEF together = 2 rt. ∠s

i. e., the ∠s ABC and DEF are supplementary.

Q. E. D.

5. Let AB and CD bisect one another at O.
Join AC and BD.



It is required to prove that AC and BD are parallel.

Proof.—In the two △s AOC and DOB

Because $\begin{cases} AO = BO \text{ (given)} \\ OC = OD \text{ (given)} \\ \text{and the } \angle\, AOC = \text{the } \angle\, BOD \text{ (Theor. 3)} \end{cases}$

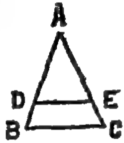
∴ the Δ^s are equal in all respects (Theor. 4)

so that, the ∠ OAC = the ∠ OBD, and these are alternate angles

∴ AC and BD are parallel (Theor. 14)

Q. E. D.

6. Let ABC be an isosceles triangle and let DE be any straight line parallel to BC.



It is required to prove that DE makes equal angles with AB and AC, *i. e.*, the ∠ ADE = the ∠ AED

Proof.—Because BC and DE are parallel and AB meets them

∴ the ext. ∠ ADE = the int. opp. ∠ ABC (Theor. 14)

Again, because BC and DE are parallel and AC meets them

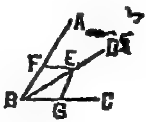
∴ the ext. ∠ AED = the int. opp. ∠ ACB (Theor. 14)

But the ∠ ABC = the ∠ ACB (Theor. 5)

∴ the ∠ ADE = the ∠ AED.

Q. E. D.

7. Let ABC be an angle and let BD be its bisector.



From E any point in BD a straight line EF is drawn parallel to BC meeting AB at F.

It is required to prove that the Δ BFE is an isosceles triangle.

Proof.—Because FE is parallel to BC and BD meets them

$\therefore$ the $\angle$ FEB=the alternate $\angle$ EBC (Theor. 14)

But the $\angle$ FBE = the $\angle$ EBC (given)

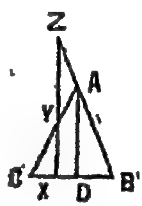
$\therefore$ the $\angle$ FEB=the $\angle$FBE

$\therefore$ FE=FB (Theor 6)

i. e., the $\triangle$ FBE is an isosceles triangle.

Similarly, by drawing EG parallel to AB it can be proved that the $\triangle$ BGE is an isosceles triangle.

8 Let ABC be an isosceles triangle. From X any point in BC, XZ is drawn perpendicular to BC, cutting AC in Y and meeting BA produced in Z.



It is required to prove that AYZ is an isosceles triangle.

The $\angle$BAC is bisected by AD meeting BC in D.

Proof—In the $\triangle^s$ ABD and ADC

Becaues { AC=AC (being sides of an isosceles triangle)
AD is common to both
and the $\angle$ BAD=the $\angle$ CAD (by construction)

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, the $\angle$ ADB=the $\angle$ ADC, and these being adjacent angles each is a rt. $\angle$.

Since BC cuts AD and ZX, and makes the int $\angle^s$ ADX and DXZ together equal to 2 rt. $\angle^s$

$\therefore$ AD and ZX are parallel (Theor. 13)

Because AD and ZX are parallel and BZ meets them

$\therefore$ the ext. $\angle$ BAD=the int. opp. $\angle$ AZX or $\angle$ZY

(Theor. 14)

Again, because DA and YZ are parallel and AY meets them

∴ the ∠ DAY = the alternate ∠ AYZ (Theor. 14)

But the ∠ BAD = the ∠ DAY or DAC (by construction)

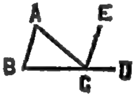
∴ the ∠ AZY = the ∠ AYZ

∴ AZ = AY (Theor 6)

i e, △ AYZ is isosceles.

Q. E. D.

9 Let CE be the bisector of the exterior angle ACD of the triangle ABC which is drawn parallel to the opposite side AB.



It is required to prove that the triangle ABC is an isosceles triangle

Proof —Because AB and CE are parallel and BD meets them

∴ the ext ∠ ECD = the int. opp. ∠ ABC (Theor. 14)

Again, because AB and CE are parallel and AC meets them

∴ the ∠ BAC = the alternate ∠ ACE (Theor. 14)

But the ∠ ECA = the ∠ ECD (given)

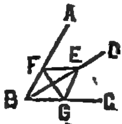
∴ the ∠ ABC = the ∠ BAC

∴ BC = AC (Theor 6)

i. e., ABC is an isosceles triangle.

Q E. D.

10 Let ABC be an angle and let BD be its bisector. From E any point in BD the straight lines EF and EG are drawn parallel to BC and AB respectively meeting AB, BC at F and G.



It is required to prove that FE and EG are equal and that the figure FB GE is a rhombus.

Join FG.

Proof.—Because EF and BG are parallel and BE meets them

$\therefore$ the $\angle$ FEB = the alternate $\angle$ EBG (Theor. 14)

But the $\angle$ EBC = the $\angle$ EBA (given)

$\therefore$ the $\angle$ FEB = the $\angle$ EBF

$\therefore$ FE = FB (Theor. 6)

Similarly, it can be proved that GE = GB

Now, in the two $\triangle^s$ EFG and FBG

Because $\begin{cases} \text{EF} = \text{FB (proved)} \\ \text{GE} = \text{GB (proved)} \\ \text{and FG is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 7)

so that, the $\angle$ FGE = the $\angle$ BGF

But the $\angle$ FGE = the alternate $\angle$ BFG (Theor. 14)

$\therefore$ the $\angle$ BGF = $\angle$ BFG

$\therefore$ BG = BF (Theor. 6)

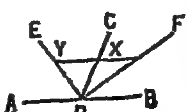
But BF = FE and BG = GE

$\therefore$ BF = GF = BG = FE

$\therefore$ the figure FBGE is a rhombus.

Q. E. D.

11. Let AB be a straight line and let CD be any straight line intersecting AB at D. DE and DF bisect the $\angle^s$ ADC and CDB respectively. Through X any point in DC, YXZ is drawn parallel to AB and terminated by DE and DF at Y and Z.



It is required to prove that YX and XZ are equal.

Proof—Because YX and AD are parallel and YD meets them

∴ the ∠ XYD=the alternate ∠ YDA (Theor. 14)

But the ∠ ADY=the ∠ YDX

∴ the ∠ YDX=the ∠ XYD

∴ XD=YX (Theor 6)

Similarly, it can be proved that XD=XZ

∴ YX=XZ

Q E D.

12 Two straight rods PA and QS start parallel and pointing the same way, and PA revolves more rapidly than QB Then they will be again parallel

(i) pointing opposite ways when PA has made half a revolution more than QB,

and (ii) pointing the same way when PA has made one complete revolution more than QB

Now, PA makes 12 complete revolutions in a minute and QB makes 10 complete revolutions in a minute.

∴ PA makes 2 complete revolutions more than QB in 1 min

(i) ∴ PA makes $\frac{1}{2}$ revolution more than QB in $\frac{1}{4}$ min. or 15 sec

(ii) PA makes 2 complete revolutions more than QB in 1 min.

∴ PA makes 1 complete revolution more than QB in $\frac{1}{2}$ min. or 30 sec.

∴ The two rods PA and QB will be again parallel,

(i) pointing opposite ways after 15 sec and (ii) pointing the same way after 30 sec., when they start parallel and pointing the same way.

Page 43.

1. We know that three angles of every triangle are together equal to 2 rt. $\angle^s$ (Theor. 16)

We also know that in an equilateral triangle all the three angles are equal to one another

$\therefore$ Each angle $= \frac{1}{3}$ of 2 rt. $\angle^s$

$= \frac{1}{3}$ of $180°$

$= 60°$.

2 We know that three angles of every triangle are together equal to 2 rt. $\angle^s$ (Theor. 16)

In a right angled isosceles triangle, one angle is a right angle.

$\therefore$ the sum of the other two angles is 2 rt. $\angle^s - 1$ rt. $\angle$ or 1 rt. $\angle$.

But the equal sides of isosceles triangle subtend equal angles

$\therefore$ Each of the equal angles $= \frac{1}{2}$ of 1 rt. $\angle = 45°$.

3. We know that three angles of every triangle are together equal to 2 rt. $\angle^s$ (Theor. 16)

The sum of two angles $= 360° - 123° = 159°$

$\therefore$ the third angle $= 180° - 159° = 21°$.

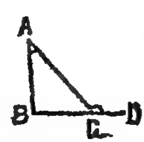
4. We know that three angles of every triangle are together equal to 2 rt. $\angle^s$ (Theor. 16)

$\therefore$ In the $\triangle$ ABC, the $\angle^s$ ACB + ABC + BAC = $180°$

But the $\angle^s$ ABC + ACB = $111° + 42° = 158°$

$\therefore$ the $\angle$ BAC $= 180° - 153° = 27°$.

5. ABC is a triangle whose side BC is produced to D.



If the exterior $\angle$ ACD be 134° and the $\angle$ BAC be 42°

It is required to find the remaining interior angles ABC and ACB

The $\angle$ ACB is supplement of the $\angle$ ACD

the $\angle$ ACD = 134°, $\therefore$ the $\angle$ ACB = 180° − 134° = 46°

Three angles of every triangle are together equal to 2 rt. $\angle^s$ (Theor 16)

$\therefore$ the $\angle$ ABC = 180° − (the $\angle^s$ BAC + ACB)

= 180° − (42° + 46°)

= 180° − 88°

= 92°.

6. (See Fig. in Ex 5).

ABC is a triangle whose side BC is produced to D

Let the exterior $\angle$ ACD be 118° and the $\angle$ ABC be 51°.

It is required to find the $\angle^s$ BAC and ACB.

The $\angle$ ACB is supplement of the $\angle$ ACB

But the $\angle$ ACD = 118°, $\therefore$ the $\angle$ ACB = 180° − 118° = 62°

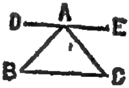
The sum of the $\angle^s$ ACB and ABC = 62° + 51° = 113°

Three angles of every triangle = 2 rt. $\angle^s$.

$\therefore$ the $\angle$ BAC = 180° − 113°

= 67°.

7. Let ABC be a triangle and let DAE be parallel to the base BC drawn through the vertex A.



It is required to prove that the three angles of the triangle ABC are together equal to 2 rt. $\angle^s$

Proof.—Because DE and BC are parallel and AC meets them

$\therefore$ the $\angle$ EAC = the alternate $\angle$ ACB (Theor. 14)

Again, because DE and BC are parallel and AB meets them

$\therefore$ the $\angle$ DAB = the alternate $\angle$ ABC (Theor. 14)

$\therefore$ the $\angle$EAC + the $\angle$ DAB = the $\angle$ ACB + the $\angle$ABC

To each of these equals add the $\angle$ BAC

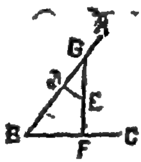
$\therefore$ the $\angle$ ABC + the $\angle$ BAC + the $\angle$ ACB

= the $\angle$ DAB + the $\angle$ BAC + the $\angle$ EAC

= 2 rt. $\angle^s$ (Theor 1)

Q. E. D.

8. Let AB, BC be any two straight lines cutting at B, and let ED, EF be two other straight lines perpendiculars to AB, BC respectively.



It is required to prove that the acute angle between AB, BC is equal to the acute angle between ED, EF.

Produce FE to meet AB in G.

Proof.—Because in the $\triangle$ GDE, the $\angle$ GDE is a rt. $\angle$

$\therefore$ the sum of the $\angle^s$ DEG and DGE is also a rt. $\angle$

i. e., the $\angle$ DGE is complement of the $\angle$ DEG (Theor. 16. Inf. 3)

Again, because in the $\triangle$ GBF, $\angle$GFB is a rt. $\angle$

$\therefore$ the sum of the $\angle^s$ GBF and BGF is also a rt. $\angle$

i. e., the $\angle$ GBF is complement of the $\angle$ BGF, or DGE (Theor. 16. Inf. 3)

$\therefore$ the $\angle$ DEG = the $\angle$ GBF (Cor. 3. Theor. 1)

= the $\angle$ ABC

Q. E. D.

Page 45.

1. Suppose ABC is a triangle in which the angle ABC is double and the angle ACB treble of the angle BAC.

It is required to find each of the angles ABC, ACB and BAC in degrees.

The $\angle$ BAC + the $\angle$ ABC + the $\angle$ ACB

$= \angle BAC + 2 \angle BAC + 3 \angle BAC$

$= 6 \angle BAC$.

But the $\angle$ BAC + the $\angle$ ABC + the $\angle$ ACB $= 180^\circ$ (Theor. 16 Inf. 1)

$\therefore 6 \angle BAC = 180^\circ$

$\therefore \angle BAC = 30^\circ$, $\angle ABC = 2 \times 30^\circ$, or 60°, and $\angle ACB = 3 \times 30^\circ$, or 90°.

2 In an isosceles triangle ACB the angles subtended by equal sides are equal (i) Each of the base angles ABC and ACB is double of the vertical angle BAC

It is required to find each of the angles ABC, ACB and BAC in degrees

the $\angle$ BAC + the $\angle$ ABC + the $\angle$ ACB

$= \angle BAC + 2 \angle BAC + 2 \angle ACB$

$= 5 \angle BAC$

But the $\angle^s$ BAC, ABC and ACB $= 180^\circ$ (Theor. 16. Inf. 1)

$\therefore 5 \angle BAC = 180^\circ$

$\therefore \angle BAC = 36^\circ$, $\angle ABC = 2 \times 36^\circ$ or 72°, and $\angle ACB = 72^\circ$

(ii) Each of the base angles ABC and ACB is four times the vertical angle BAC.

It is required to find each of the angles ABC, ACB and BAC in degrees.

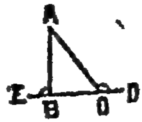
$$\angle BAC + \angle ABC + \angle ACB = \angle BAC + 4\angle BAC + 4\angle BAC$$
$$= 9 \angle BAC$$

But $\angle$ BAC + $\angle$ ABC + $\angle$ ACB = 180° (Theor. 16. Inf. 1)

$\therefore$ 9 $\angle$ BAC = 180°

or, $\angle$ BAC = 20°, $\angle$ ABC = 4 × 20°, or 80° and $\angle$ ACB = 80°

3. Let DE be a straight line. Take any two points B and C in DE. At B and C make $\angle^s$ EBA and DCA = 94° and 126° respectively, the sides BA and CA meeting in A.



It is required to find the vertical angle BAC.

Because the $\angle^s$ ABE and ABC together = 2 rt. $\angle^s$ (Theor. 1)

the $\angle$ ABE = 94°; $\therefore$ the $\angle$ ABC = 180° − 94° = 86°

the exterior $\angle$ ACD = int. opp. $\angle^s$ ABC and BAC.

the $\angle$ ADC = 126° and the $\angle$ ABC = 86° (Obs. Theor. 16)

$\therefore$ the $\angle$ BAC = 126° − 86° = 40°.

4. In a triangle ABC, the sum of the base angles ABC and ACB is 162° and their difference is 60°.

It is required to find all the angles ABC, ACB and BAC of the triangle ABC.

Suppose the $\angle$ ABC is greater than the $\angle$ ACB

Because the $\angle$ ABC + the $\angle$ ACB = 162°

and the $\angle$ ABC − the $\angle$ ACB = 60°

$\therefore$ by adding we have 2 $\angle$ ABC = 162° + 60° = 222°

$\therefore$ $\angle$ ABC = $\frac{1}{2}$. 222° = 111°

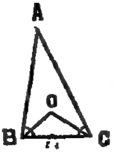
and by subtracting we have 2 $\angle$ ACB = 162° − 60° = 102°

$\therefore$ $\angle$ ACB = $\frac{1}{2}$. 102° = 51°.

Again, because the $\angle^s$ ABC + ACB + BAC = 180° (Inf. 1 Theor. 16)

$\therefore$ the $\angle$ BAC = 180° − (ABC + ACB)
= 180° − 162°
= 18°

5. Let ABC be a triangle in which the angles ABC and ACB are equal to 84° and 62°, respectively and the angles ABC and ACB are bisected by BO and CO respectively, the lines BO, CO meeting at O



It is required to find the $\angle^s$ BAC and BOC

In the $\triangle$ ABC, the $\angle^s$ ABC + ACB + BAC = 180°

(Theor. 16. Inf. 1)

the $\angle$ ABC = 84° and the $\angle$ ACB = 62°

$\therefore$ the $\angle$ BAC = 180° − ($\angle$ ABC + $\angle$ ACB)
= 180° − (84° + 62°)
= 180° − 146°
= 34°

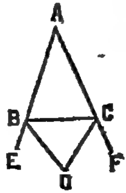
the $\angle$ OBC = $\frac{1}{2}$ $\angle$ ABC = $\frac{1}{2}$. 84° = 42°

and the $\angle$ OCB = $\frac{1}{2}$ $\angle$ ACB = $\frac{1}{2}$ 62° = 31°

In the $\triangle$OBC, the $\angle^s$ OBC + OCB + BOC = 180° (Theor. 16. Inf 1)

$\therefore$ the $\angle$ BOC = 180° − ($\angle$ OBC + $\angle$ OCB)
= 180° − (42° + 31°)
= 180° − 73°
= 107°

6. Let BC be a straight line. At B make the angle CBA = 74° and at C make the angle BCA = 62°, the sides BA, CA meeting in A Produce AB, and AC to any points E and F respectively. Let the bisectors BO, CO of the exterior angles CBE and BCF meet in O.



It is required to find the $\angle$ BOC.

the $\angle^s$ ABC and CBE = 2 rt. $\angle^s$ = 180° (Theor. 1)

But the $\angle$ ABC = 74°, $\therefore$ the $\angle$ CBE = 180° − 74° = 106°

the, $\angle$ CBO $= \frac{1}{2} \angle$ CBE $= \frac{1}{2} \times 106^\circ = 53^\circ$

also, the $\angle^s$ ACB and BCF $= 2$ rt. $\angle^s = 180^\circ$ (Theor. 1)

But the $\angle$ ACB $= 62^\circ$. $\therefore$ the $\angle$ BCF $= 180^\circ - 62^\circ = 118^\circ$

$\therefore$ the $\angle$ BCO $= \frac{1}{2} \angle$ BCF $= \frac{1}{2} \times 118^\circ = 59^\circ$.

In the $\triangle$ BCO, the $\angle^s$ BCO + OBC + BOC $= 180^\circ$

$\therefore$ the $\angle$ BOC $= 180^\circ - (\angle$ BCO $+ \angle$ CBO) (Theor. 16. Inf. 1)

$$= 180^\circ - (59^\circ + 53^\circ)$$
$$= 180^\circ - 112^\circ$$
$$= 68^\circ.$$

7. In a quadrilateral the sum of all the angles is 4 rt. $\angle^s$, because by joining any of its diagonal, the figure (quadrilateral) is divided into two triangles and we know the sum of all angles of a triangle to be 2 rt. $\angle^s$ (Theor 16).

the sum of three angles $= 114\frac{1}{2}^\circ + 50^\circ + 75\frac{1}{2}^\circ = 240^\circ$
the sum of four angles of a quadrilateral $= 4$ rt. $\angle^s = 360^\circ$

$\therefore$ the fourth angle $= 360^\circ - 240^\circ = 120^\circ$.

8 In a quadrilateral ABCD, the $\angle$ ABC $= 2 \angle$ BAD, the $\angle$ DCB $= 3 \angle$ BAD and the $\angle$ ADC $= 4 \angle$ BAD.

the $\angle$ BAD $+ \angle$ ABC $+ \angle$ DCB $+ \angle$ ADC

$= \angle$ BAD $+ 2 \angle$ BAD $+ 3 \angle$ BAD $+ 4 \angle$ BAD

$= 10 \angle$ BAD

But the $\angle$ BAD $+ \angle$ ABC $+ \angle$ DCB $+ \angle$ ADC $= 360^\circ$

$\therefore 10 \angle$ BAD $= 360^\circ$

$\therefore \angle$ BAD $= 36^\circ$. $\angle$ ABC $= 2 \times 36^\circ = 72^\circ$;

$\angle$ DCB $= 3 \times 36^\circ = 108^\circ$, and $\angle$ ADC $= 4 \times 36^\circ = 144^\circ$

9. All the interior angles of a pentagon + 4 rt. $\angle^s = 10$ rt. $\angle^s$ (Cor. 1. Theor. 16)

$\therefore$ all the interior angles of the pentagon $= 10$ rt. $\angle^s - 4$ rt. $\angle^s = 6$ rt. $\angle^s = 540^\circ$.

the sum of four angles of the pentagon $=40^\circ+78^\circ+122^\circ+135^\circ=375^\circ$

$\therefore$ the fifth angle $=540^\circ-375^\circ=165^\circ$

10. It is required to prove that in any regular polygon of n sides each angle contains $\frac{2(n-2)}{n}$ right angles.

(*i*) Because all the interior angles of any rectilineal figure of n sides $+4$ rt $\angle^s=2n$ rt $\angle^s$ (Cor. 1 Theor. 16)

$\therefore$ all the interior angles $=2n$ rt $\angle^s-4$ rt $\angle^s$

$=2(n-2)$ rt $\angle^s$

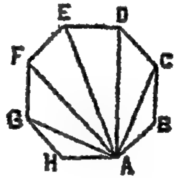
We know that in any figure there are as many angles as there are sides

$\therefore$ a figure containing n sides has n angles

$\therefore$ n angles $=2(n-2)$ rt $\angle^s$

$\therefore$ each angle $=\frac{2(n-2)}{n}$ rt. $\angle^s$

(*ii*) Let ABCDEFGH be a regular polygon of n sides and hence having all its angles equal.



By joining one vertex A to each of the others (except the two immediately adjacent to A), that is, by joining AC, AD, AE, AF, ..., the figure will be divided into $(n-2)$ triangles

Because in every triangle the sum of three angles $=2$ rt. $\angle^s$ Theor. 16)

$\therefore$ all the angles of $(n-2)$ triangles $=2(n-2)$ rt $\angle^s$

i. e, all the angles of the polygon ABCDEFGH

$=2(n-2)$ rt. $\angle^s$.

We know that there are as many angles as the figure has sides.

$\therefore$ a polygon of n sides has n angles; and the value of n angles $=2(n-2)$ rt. $\angle^s$

$\therefore$ each angle of the polygon ABCDEFGH $=\dfrac{2(n-2)}{n}$ rt. angles.

11. It is required to find the number of sides in the regular polygons each of whose angles is (*i*) 108°, and (*ii*) 156°.

Because $nD+360°=n.\ 180°$, where D denotes the number of degrees in an angle of a regular polygon of n sides (Theor. 16. Cor. 1)

$(i)\ \therefore\ n.108°+360°=n.180°$

or, $n\,(180°-108°)=360°$

or, $n\ 72°=360°$

$\therefore\ n=\dfrac{360°}{72°}=5.$

i. e., the figure has 5 sides.

and $(ii)\ n\ 156°+360°=n\ 180°$

or $n.\ 180°-156°)=360°$

or $n.\ 24°=360°$

$\therefore\ n=\dfrac{360°}{24°}=15$

i. e., the figure contains 15 sides.

12 Regular figures can be fitted together so as to form a plane surface only when the sum of the consecutive angles formed at any point within that plane by placing them together is equal to four right angles.

But since each of the regular figures has the same number of sides, therefore the consecutive angles so formed at that point are all equal to one another.

∴ each of these consecutive angles must be a factor of four right angles, or 360°.

Since a (regular) figure must have at least three sides, therefore the least value of an angle of a regular polygon is 60°.

Also the angle of a regular polygon must in every case be less than 180°

∴ the magnitude of the angles of all such polygons lie between 60 and 180°, including the former and excluding the latter.

Again, because the factors of 360° lying between 60° and 180° (including 60°, and excluding 180°) are 60°, 72°, 90° and 120°, and of these factors 72° is not the value of an angle of any regular polygon

∴ the regular figures, which can be fitted together so as to form a plane surface, must have the value of their angles 60°, 90° or 120°, that is, they must be equilateral triangles, squares, or regular hexagons.

Page 47.

1. Because all the exterior angles of any rectilineal figure = 4 rt $\angle^s$ (Theor. 16, Cor 2)

and all the angles of a regular polygon are equal

∴ each of the exterior angles of a regular polygon of six sides $= \frac{4}{6}$ of a rt. $\angle = \frac{4}{6}$ of 90°, or 60°.

But each of the interior angles of an equilateral triangle

$$= \frac{180^\circ}{3} = 60^\circ$$

∴ the ext. angle of a regular hexagon = the int. angle of an equilateral triangle

2. Because all the exterior angles of any rectilineal figure = 4 rt $\angle^s$ (Theor. 16 Cor. 2)

and all the angles of a regular polygon are equal

(i) ∴ Each of the exterior angles of a regular octagon (of eight sides) $= \frac{4}{8}$ of a rt. $\angle = \frac{4}{8}$ of 90°, or 45°

and, (*ii*) each of the exterior angles of a regular decagon (10 sides)$=\frac{4}{10}$ of a rt $\angle=\frac{4}{10}$ of $90°$, or $36°$.

3. Suppose the figure contains n sides, then there are n equal exterior angles (because the figure is regular)

(*i*) the value of each ext. angle$=30°$

$\therefore$ n. $30°=4$ rt. $\angle^s=360°$

$\therefore n=\dfrac{360°}{30°}=12$.

(*ii*) the value of each ext. angle$=20°$

$\therefore$ n. $24°=4$ rt. $\angle^s=360°$

$\therefore n=\dfrac{360}{24}=15$.

4. Let AO meet two parallel straight lines AB, OD and let the two interier angles BAO and AOD on the same side of AO be bisected by AE and OE respectively, meeting at E.

It is required to show that the $\angle$ AEO is a rt. $\angle$.

Proof.—Because AB and OD are parallel and AO meets them

$\therefore$ the int. $\angle^s$ BAO and AOD$=2$ rt. $\angle^s$ (Theor. 14)

the $\angle$ OAE$=\frac{1}{2}$ of the $\angle$ BAO; and the $\angle$ AOE$=\frac{1}{2}$ of the $\angle$ AOD

$\therefore$ the $\angle^s$ OAE and AOE$=\frac{1}{2}$ of $\angle^s$ BAO and AOD

$=\frac{1}{2}$ of 2 rt. $\angle^s$

$=1$ rt. $\angle$.

In any triangle the sum of the three angles$=2$ rt $\angle^s$ (Theor. 16)

$\therefore$ in the $\triangle$ AOE, the $\angle$ AEO$=2$ rt. $\angle^s-(\angle^s$ OAE $+$ AOE)

$=2$ rt $\angle^s-1$ rt. $\angle$

$=1$ rt. $\angle$

i. e., the bisectors of the $\angle^s$ BAO and AOD meet at right angles.

Q. E. D.

5. (See Fig. in Ex. 3 on p. 45).

Let ABC be a triangle whose base BC is produced bothways to points D and E.

It is required to prove that the exterior $\angle^s$ ACD + ABE—the vertical angle BAC = 2 rt. $\angle^s$

Proof.—The ext. $\angle$ACD = the int. $\angle^s$ ABC + BAC

(Obs. Theor. 16)

also, the ext. $\angle$ ABE = the int. $\angle^s$ ACB + BAC

(Obs. Theor. 16)

By adding we have

the $\angle^s$ ACD + ABE = $\angle^s$ ABC + BAC + ACB + BAC

or, $\angle$ACD + $\angle$ABE − $\angle$ BAC = $\angle$ ABC + $\angle$ BAC + $\angle$ACB

= 2 rt. $\angle^s$ (Theor. 16)

i.e., the ext. $\angle^s$ ACD + ABE − the vertical angle BAC = 2 rt. $\angle^s$.

Q. E. D.

6. (See Fig. in Ex. 5 on p. 45)

Let ABC be a triangle and let the base angles ABC and ACB be bisected by BO and CO meeting at O.

It is required to show that the $\angle$ BOC $= 90° + \frac{\angle \text{BAC}}{2}$.

Proof.—the $\angle$ OBC $= \frac{\angle \text{ABC}}{2}$, and the $\angle$ OCB $= \frac{\angle \text{ACB}}{2}$

In the $\triangle$ OBC, the $\angle^s$ OBC + OCB + BOC = 180°.

(Inf. 1 Theor. 16)

or, the $\angle$ BOC $+ \frac{\angle \text{ABC}}{2} + \frac{\angle \text{ACB}}{2} = 180°$...(1)

In the $\triangle$ ABC, the $\angle^s$ ABC + ACB + BAC = 180°

(Inf. 1, Theor. 16)

$$\therefore \frac{\angle ABC}{2}+\frac{\angle ACB}{2}+\frac{\angle BAC}{2}=\tfrac{1}{2}.\ 180^\circ=90^\circ \ldots(2)$$

Subtracting (2) from (1) we have

$$\angle\ BOC-\frac{\angle BAC}{2}=180^\circ-90^\circ=90^\circ$$

$$\therefore\ \angle\ BOC=90^\circ+\frac{\angle BAC}{2}$$

Q. E. D.

7. (See fig. in Ex. 6 on p. 45).

Let ABC be a triangle whose sides AB and AC are produced to any points E and F respectively. Let the exterior angles CBE and BCF be bisected by BO and CO meeting at O.

It is required to show that the angle $BOC=90^\circ-\dfrac{\angle BAC}{2}$

Proof—In the $\triangle$ OBC, the $\angle^s$ $OBC+BOC+OCB=180^\circ$

$\therefore 2\angle OBC+2\angle OCB+2\angle BOC=360^\circ$ [(Theor. 16)

or, $\angle CBE+\angle BCF+2\angle BOC=360^\circ \ldots(1)$

$\angle EBC+\angle ABC+\angle BCF+\angle BCA=360^\circ \ldots(2)$

subtracting (2) from (1), we have [Theor. 1)

$2\angle BOC-\angle ABC$—the $\angle ACB=0$

or, $2\angle BOC=\angle ABC+\angle ACB$

In the $\triangle$ ABC, the $\angle ABC+$the $\angle BAC+$the $\angle ACB$ $=180^\circ$ (Int. 1. Theor. 16)

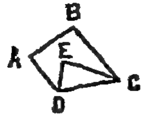
$\therefore ABC+\angle ACB=180^\circ-\angle BAC$

$\therefore 2\angle BOC=180^\circ-$the $\angle BAC$

or, $\angle\ BOC=90^\circ-\dfrac{\angle BAC}{2}$

Q. E. D.

8. Let ABCD be any quadrilateral and let any two consecutive angles ADC and BCD be bisected by DE and CE respectively, meeting at E.



It is required to show that the angle $DEC = \frac{1}{2}(\angle DAB + \angle ABC)$

Proof—In the $\triangle$ DEC, $\angle^s DEC + EDC + ECD = 180^\circ$
(Theor. 16. Inf. 1)

$\therefore 2 \angle DEC + 2 \angle EDC + 2 \angle ECD = 360^\circ$

or, $2 \angle DEC + \angle ADC + \angle BCD = 360^\circ$ ..(1)

In the quadrilateral ABCD, $\angle ABC + \angle BCD + \angle CDA + \angle DAB = 360^\circ$.. (2)

Subtracting (2) from (1), we have

$2 \angle DEC - \angle DAB - \angle ABC = 0$

or, $2 \angle DEC = \angle DAB + \angle ABC$

$\therefore \angle DEC = \frac{1}{2} (\angle DAB + \angle ABC)$.

Q. E. D.

9. Let ABC be an isosceles triangle whose vertex is A and whose equal sides are AB, AC, and the side BA is produced to any point D making AD equal to BA. Join DC.

It is required to show that BCD is a right angle.

Proof—Because AB=AC (given)

$\therefore \angle ABC = \angle ACB$ (Theor. 5)

Again, because AC=AD

$\therefore \angle ACD = ADC$ (Theor. 5)

$\therefore \angle ACD + \angle ACB =$ the $\angle DBC +$ the $\angle BDC$

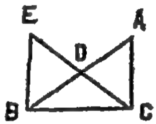
$= \frac{1}{2}$ of 2 rt. $\angle^s$

$= 1$ rt. $\angle$

or, the $\angle BCD$ is a rt. $\angle$

Q. E. D.

10 Let ABC be a right-angled triangle, right angled at C. The hypotenuse AB is bisected at D and CD is joined.



It is required to prove that $CD = \frac{1}{2} AB$.

Produce CD to any point E making $DE = CD$. Join BE.

Proof—In the two $\triangle^s$ ADC and DEB

Because $\begin{cases} AD = DB \text{ (given)} \\ DC = DE \text{ (by construction)} \\ \text{and the } \angle ADC = \text{the } \angle EDB \quad \text{(Theor. 3)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, the $\angle DAC =$ the $\angle EBD$, and $AC = BE$

In the $\triangle ABC$, the $\angle ACB$ is a rt $\angle$.

$\therefore$ the $\angle^s BAC + ABC = 1$ rt. $\angle$ (Inf. 3. Theor. 16)

$\therefore$ the $\angle^s EBD + ABC = 1$ rt. $\angle$

or, the $\angle EBC$ is a rt $\angle$

In the $\triangle^s$ ABC and EBC

Because $\begin{cases} AC = BE \text{ (proved)} \\ BC \text{ is common to both} \\ \text{and the } \angle ACB = \text{the } \angle EBC \text{ (being rt. } \angle^s) \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

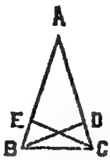
so that, $AB = EC$

But $DE = \frac{1}{2} EC$, $\therefore DC = \frac{1}{2} AB$.

Q. E. D.

Page 49

1 Let ABC be an isosceles triangle whose equal sides are AB and AC From B and C perpendiculars BD and CE are drawn to AC and AB respectively.



It is required to prove that BD and CE are equal.

Proof—In the $\triangle$ ABC, because AB=AC

$\therefore$ the $\angle$ ABC=the $\angle$ ACB (Theor 5)

In the two $\triangle^s$ BDC and EBC

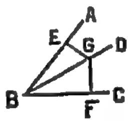
Because $\begin{cases} \text{the } \angle \text{ BDC=the } \angle \text{ CEB (being rt } \angle^s) \\ \text{the } \angle \text{ DCB=the } \angle \text{ EBC (proved)} \\ \text{and BC is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects. (Theor. 17)

so that, BD=CE.

Q E D.

2 Let ABC be an angle and let BD be its bisector From G any point in BD perpendiculars GE and GF are drawn to AB and BC respectively.



It is required to prove that EG and GF are equal.

Proof—In the two $\triangle^s$ EBG and BGF

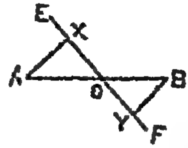
Because $\begin{cases} \text{the } \angle \text{ GEB=the } \angle \text{ GFB (being rt } \angle^s) \\ \text{the } \angle \text{ EBG=the } \angle \text{ GBF (given)} \\ \text{and BG is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, EG=GF.

Q. E. D.

3. Let AB be a straight line whose middle point is O, and let EOF be another straight line drawn through O From A and B perpendiculars AX, BY are drawn on EF.



It is required to prove that AX and BY are equal.

Proof.—In the two $\triangle^s$ AXO and BYO

Because $\begin{cases} \text{the } \angle \text{ AXO} = \text{the } \angle \text{ BYO (being rt. } \angle^s) \\ \text{the } \angle \text{ XOA} = \text{the } \angle \text{ BOY (Theor. 3)} \\ \text{and AO} = \text{BO (given)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, AX = BY.

4. (See Fig. in Ex. 1 on p. 19).

Let ABC be a triangle and let the bisector AD of the vertical angle BAC meet the base BC at right angles in D.

It is required to prove that ABC is an isosceles triangle.

Proof—In the two $\triangle^s$ ABD and ACD

Because $\begin{cases} \text{the } \angle \text{ ADB} = \text{the } \angle \text{ ADC (being rt } \angle^s) \\ \text{the } \angle \text{ BAD} = \text{the } \angle \text{ CAD (given)} \\ \text{and AD is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, AB = AC

$\therefore$ the $\triangle$ ABC is isosceles.

Q. E. D.

5. (See Fig. in Ex. 1 on p. 19).

Let ABC be a triangle and let the perpendicular AD drawn from the vertex A bisect the base BC.

It is required to prove that ABC is an isosceles triangle.

Proof.—In the two $\triangle^s$ ABD and ACD

Because $\begin{cases} BD = DC \text{ (given)} \\ AD \text{ is common to both} \\ \text{and the } \angle\, ADB = \text{the } \angle\, ADC \text{ (being rt. } \angle^s) \end{cases}$

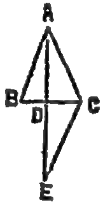
$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, $AB = AC$.

$\therefore$ the $\triangle$ ABC is isosceles.

Q. E. D.

6 Let ABC be a triangle and let AD the bisector of the vertical angle BAC bisect the base BC.



It is required to prove that ABC is an isosceles triangle.

Produce AD to any point E making $DE = AD$. Join EC.

Proof—In the two $\triangle^s$ ABD and DEC

Because $\begin{cases} AD = DE \text{ (by construction)} \\ BD = DC \text{ (given)} \\ \text{and the } \angle\, ADB = \text{the } \angle\, CDE \quad \text{(Theor. 3)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, $AB = CE$, and the $\angle$ BAD = the $\angle$ DEC

But the $\angle$ BAD = the $\angle$ CAD (given)

$\therefore$ the $\angle$ CAD = the DEC

or, the $\angle$ CAE = the $\angle$ AEC

$\therefore$ $AC = CE$ (Theor. 6)

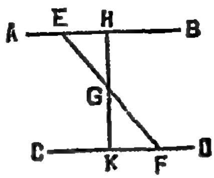
But $AB = CE$ (proved)

$\therefore$ $AB = AC$

$\therefore$ the $\triangle$ ABC is isosceles.

Q. E. D.

7 Let the straight line EF meet two parallel straight lines AB, CD and be terminated by them at E and F. Let G be the middle point of EF.



It is required to prove that G is equidistant from AB and C D.

From G draw GH perpendicular to BA and produce HG to meet CD in K. Then HK is also perpendicular to CD.

Proof—In the two $\triangle^s$ HGE and GFK

Because $\begin{cases} \text{EG} = \text{GF (given)} \\ \text{the } \angle \text{ HGE} = \text{the } \angle \text{ FGK (Theor. 3)} \\ \text{and the } \angle \text{ EHG} = \text{the } \angle \text{ GKH(being rt. } \angle^s) \end{cases}$

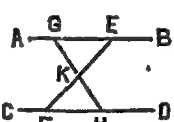
$\therefore$ two $\triangle^s$ are equal in all respects (Theor 17)

so that, HG=GK

$\therefore$ G is equidistant from AB and CD.

Q. E. D.

8. Let the straight line EF be drawn between two parallel straight lines AB, CD and be terminated by them. Let the straight line EF be bisected at K, and let GKH be another straight line drawn through K and terminated by the parallel straight lines.



It is required to prove that GK=KH

Proof—In the two $\triangle^s$ GKE and FKH

Because $\begin{cases} \text{the } \angle \text{ GKE} = \text{the } \angle \text{ FKH (Theor. 3)} \\ \text{the } \angle \text{ EGK} = \text{the alt. } \angle \text{ KHF (Theor. 14)} \\ \text{and EK} = \text{KF (given)} \end{cases}$

$\therefore$ two Δ^s are equal in all respects (Theor 17)

so that, GK=KH

i. e, the straight line GH passing through K the middle point of EF and terminated by the parallel straight lines AB, CD is bisected at K.

Q E D

9. (See. Fig in Ex. 8)

Let AB, CD be two parallel straight lines, let EF be any straight line terminated by the parallel straight lines and let K be its middle point.

Then K is equidistant from the two parallel straight lines AB, CD.

Let GKH be another straight line drawn through K and terminated by the parallels.

It is required to prove that GE=FH.

Proof —In the two Δ^s GKE and KFH

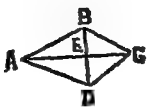
$$\text{Because} \begin{cases} KE=KF \\ \text{the } \angle\ GKE=\text{the } \angle\ FKH \quad (\text{Theor } 3) \\ \text{and the } \angle\ GEK=\text{the alt. } HFK \quad (\text{Theor } 14) \end{cases}$$

$\therefore$ two Δ^s are equal in all respects (Theor. 17)

so that, GE=FH.

Q. E. D.

10 Let ABCD be a quadrilateral in which AB=AD and BC=CD Join AC and BD and let them cut at E



It is required to prove that (*i*) AC bisects the angles BAD and BCD, and (*ii*) AC is perpendicular to BD.

Proof.—(*i*) In the two Δ^s ABC and ADC

Because $\begin{cases} AB = AD \text{ (given)} \\ BC = CD \text{ (given)} \\ \text{and } AC \text{ is common to both} \end{cases}$

$\therefore$ the two $\triangle^s$ are equal in all respects (Theor. 7)

so that, the $\angle$ BAC = the $\angle$ CAD, and the $\angle$ ACB = the $\angle$ ACD

i e., the $\angle^s$ BAD and BCD are bisected by AC.

(ii) In the two $\triangle^s$ ABE and AED

Because $\begin{cases} AB = AD \text{ (given)} \\ AE \text{ is common to both} \\ \text{and the } \angle BAE = \text{the } \angle DAE \text{ (proved)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

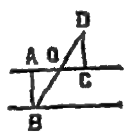
so that the $\angle$ AEB = the $\angle$ AED, and these being adjacent angles each is a right angle.

$\therefore$ AE is perpendicular to BD

i.e , AC is perpendicular to BD.

Q. E. D.

11. Let A be a point on the bank of a river and let B be an object immediately opposite to A on the other bank. Join AB Then AB indicates the breadth of the river.



From A draw a straight line AC at right angles to AB, and let O be the middle point of AC.

Join BO

From C draw CD perpendicular to AC meeting BO produced in D, Then D represents the point from which O and B are seen in the same direction.

It is required to prove that CD is equal to the breadth of the river.

Proof —In the two $\triangle^s$ DOC and AOB

Because $\begin{cases} OC = AO \text{ (given)} \\ \text{the } \angle DOC = \text{the } \angle AOB \text{ (Theor 3)} \\ \text{and the } \angle DCO = \text{the } \angle BAO \text{ (being rt. } \angle^s) \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, CD=AB

Hence CD is equal to the breadth of the river.

Q. E. D.

Page 54.

1. (*i*) It is required to state the properties of a triangle relating to the sum of its interior angles

The sum of all interior angles of every triangle=2 rt. $\angle^s$ (Theor 16)

(*ii*) It is required to state the properties of a triangle relating to the sum of its exterior angles.

If the sides of triangle be produced successively in the same direction, then the sum of all its exterior angles thus formed is equal to 4 rt. angles (Cor. 2. Theor. 16)

In a polygon of n sides the property corresponding to (*i*) is that the sum of all the interior angles together with four right angles=2 n rt $\angle^s$ (Cor. 1 Theor 16)

The triangle shares the property (*ii*) with every other rectilineal figure.

2 It is required to classify triangles with regard to their angles.

With regard to angles, the triangles are divided into (*i*) acute-angled triangles, (*ii*) right angled triangles, and (*iii*) obtuse-angled triangles.

Assumption made in this classification is that every triangle must have at least two acute-angles.

(Theor. 8. Cor. 2)

3. It is required to enunciate two theorems in which from data relating to the sides a conclusion is drawn relating to the angles.

If two sides of a triangle are equal to one another, then the angles opposite to the equal sides are equal to one another (Theor. 5)

If one side of a triangle is greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less (Theor. 9)

In the $\triangle$ ABC, because $a = c = 3{\cdot}6$ cm.
$\therefore$ the $\angle$ A = the $\angle$ C (Theor. 5)
and because a or c is greater than b
$\therefore$ the $\angle$ A or the $\angle$ C is greater than the $\angle$ B.

(Theor. 9)

$\therefore$ the $\angle$ B is the least angle

Again, because the $\angle$ A = the $\angle$ C, therefore each of them must be an acute angle, and because the $\angle$ B is less than the $\angle$ A or the $\angle$ C, therefore the $\angle$ B is also an acute angle.

$\therefore$ ABC is an acute angled triangle.

4. It is required to enunciate two theorems in which from data relating to the angles a conclusion is drawn relating to the sides.

If two angles of a triangle are equal to one another, then the sides opposite to the equal angles are equal to one another

(Theor 6)

If one angle of a triangle is greater than another, then the side opposite to the greater angle is greater than the side opposite to the less (Theor. 10)

(i) $A = 48^\circ$ and $B = 51^\circ$, $\therefore$ $A + B = 48 + 51^\circ = 99^\circ$

$\angle^s A + B + C = 180^\circ$ (Theor 16. Inf 1)

$\therefore$ $C = 180^\circ - (A + B) = 180^\circ - 99^\circ = 81^\circ$

Because C is the greatest angle, hence c is the greatest side.

(ii) $A=B=62\frac{1}{2}°$, $\therefore$ $A+B=62\frac{1}{2}°+62\frac{1}{2}°=125°$

$A+B+C=180°$ (Theor 36 Inf. 1)

$\therefore$ $C=180°-(A+B)=180°-125°=55°$

$\therefore$ C is the least angle and c is the least side (Theor. 10)

$A=B$, $\therefore$ $a=b$. (Theor 6)

Hence the arrangement of the sides in order of their lengths is a, b, c.

5 In the equality of triangles *ambiguity* arises when two sides of one triangle is equal to two sides of another triangle, and the angles opposite to shorter pair of equal sides are also given equal

(i) Because $\angle A=\angle A'=71°$, $\angle B=\angle B'=46°$ and $a=a'=3.7$ cm.

$\therefore$ we have two angles A and B of the triangle ABC equal to two angles A′ and B′ of the triangle A′B′C′, each to each, and the side a of the first equal to the corresponding side a' of the other, hence the $\triangle^s$ ABC and A′B′C′ are equal in all respects (Theor. 17)

The $\angle^s$ $A+B=71°+46°=117°$

The $\angle^s$ $A+B+C=180°$ (Theor 16 Inf 1)

$\therefore$ the $\angle$ $C=180°-(\angle^s A+B)$

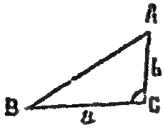
$=180°-117°=63°$

Construction—Take a straight line $BC=3.7$ cm.

At B and C make the $\angle^s$ CBA and BCA equal to 46° and 63° respectively, the arms BA and CA of the angles meeting at A Then ABC is the required triangle.

The $\triangle$ A′B′C′ can similarly be constructed.

(ii). Because $a=a'=4 \cdot 2$ cm., $b=b'=2 \cdot 4$ cm. and $\angle C = \angle C'=81^\circ$



$\therefore$ we have two sides a and b of the triangle ABC equal to two sides a' and b' of the triangle A'B'C', each to each, and the included angle C of the first equal to the included angle C' of the other, hence two $\triangle^s$ ABC and A'B'C' are equal in all respects (Theor. 4)

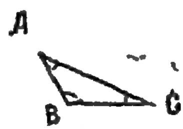
Construction—Take a straight line BC=4.2 cm At C make the $\angle$ BCA=81° making the arm AC=2·4 cm.

Join AB

Then ABC is the required triangle.

Similarly the triangle A'B'C' can be constructed.

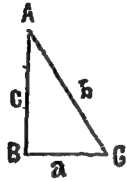
(iii). Because A=A'=36°, B=B'=121° and C=C'=23°.



$\therefore$ we have three angles A, B, C of the triangle ABC equal to three angles A', B', C' of the triangle A'B'C', each to each, hence the two triangles ABC and A'B'C' are either identically equal or similar.

Construction—Take a straight line BC of any length. At B and C make the $\angle^s$ CBA and BCA=121° and 23° respectively, the arms BA, CA meeting at A. Then ABC is the required triangle Similarly the triangle A'B'C' can be constructed taking B'C' equal to BC when the triangles will be identically equal, or making B'C' greater or less than BC when the triangles will be similar.

(*iv*) Because $a=a'=3$ cm., $b=b'=5{\cdot}2$ cm. and $c=c'=4{\cdot}5$ cm.

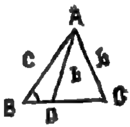


We have three sides a, b, c of the triangle ABC equal to three sides a', b', c' of the triangle A′B′C′, each to each, hence the two triangles are equal in all respects (Theor. 7)

Construction—Take a straight line BC=3 cm With centres B and C and radii equal to 4 5 cm and 5 2 cm draw two arcs cutting at A Join AB and AC. Then ABC is the required triangle,

Similarly the $\triangle$ A′B′C′ can be constructed.

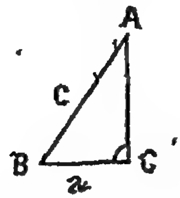
(*v*) Because $b=b'=4{\cdot}3$ cm. $c=c'=5{\cdot}0$ cm. and $\angle B=\angle B'=53°$.



$\therefore$ We have two sides b, c of the triangle ABC equal to two sides b', c' of the triangle A′B′C′, and the angle B opposite to shorter side b of the first equal to the corresponding angle B′ opposite to shorter side b' of the other; hence the ambiguity arises.

Construction—Take a straight line BC of any convenient length. At B make the $\angle$ CBA=53° making the arm BA=5 cm. With centre A and radius=4 3 cm. draw an arc cutting BC in two points C and D on the same side of B. Hence there are two triangles ABD and ABC which satisfy the given conditions Similary we can get the triangles A′B′C′ and A′B′D′ satisfying the given conditions.

(*vi*) Because $\angle C = \angle C' = 90°$, $c = c' = 5$ cm. and $a = a' = 3$ cm.



$\therefore$ We have two right-angled triangles ABC, A′B′C′ right-angled at C and C′, and one side a and the hypotenuse c of the $\triangle$ ABC respectively equal to one side a′ and the hypotenuse c′ of the $\triangle$ A′B′C′, hence two triangles are equal in all respects (Theor. 18)

Construction—Take a straight line BC=3 cm.

At C draw CA perpendicular to BC. With centre B and radius=5 cm, draw an arc cutting CA in A. Join AB. Then ABC is the required triangle

Similarly the $\triangle$ A′B′C′ can be constructed.

6. (*i*) It is required to state generally under what conditions two triangles are necessarily congruent.

Two triangles are necessarily congruent in any of the following cases :—

(1) When two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other.

(2) When two triangles have three sides of the one equal to three sides of the other.

(3) When two triangles have two angles and any one side of the one equal to two angles and the corresponding side of the other.

(4) When two triangles are right-angled, and one side and hypotenuse of the one equal to the corresponding side and hypotenuse of the other.

(*ii*) It is required to state generally under what conditions triangles may or may not be congruent.

Two triangles may or may not be congruent in any of the following cases —

(1) When two triangles have three angles of the one equal to three angles of the other.

(2) When two triangles have two sides and the angle opposite to shorter given side of the one equal to two sides and the angle *opposite to* the shorter given side *of* the other.

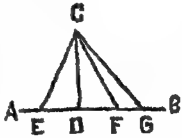
7. It is required to explain carefully that if two triangles have their angles equal, each to each, the triangles are not necessarily equal in all respects because the three data are not independent.

If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one *must* be equal to the third angle of the other (Theor. 16. Inf 2)

Thus the third relation is only a consequence of the first two Hence the three data are not independent Without some further data we cannot conclude that the triangles are equal in all respects

Page 55.

8. Let AB be a straight line and C be any external point. From C draw CD perpendicular to AB. Let CE and CF be any two obliques making equal angles ECD, FCD with the perpendicular CD Let GC be any other oblique such that the ∠ DCE is less than the ∠ DCG



(1) It is required to prove that the perpendicular CD is the shortest line.

Proof.—In the △ ECD, the ∠ EDC is a rt angle

∴ the ∠ CED is an acute angle (Theor 8 Cor. 1)

∴ the ∠ EDC is greater than the ∠ CED

∴ CE is greater than CD (Theor. 10)

Similarly it can be proved that any other oblique drawn from C to AB is greater than CD.

$\therefore$ the perpendicular CD is the shortest line

(*ii*) It is required to prove that the obliques CE and CF are equal.

Proof —In the two $\triangle^s$ CED and CDF

because $\begin{cases} \text{the } \angle \text{ ECD} = \text{the } \angle \text{ DCF (given)} \\ \text{the } \angle \text{ EDC} = \text{the } \angle \text{ CDF being rt. } \angle^s\text{)} \\ \text{and CD is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, CE $=$ CF.

(*iii*) It is required to prove that CF is less than CG.

In the two triangles EDC and CDF, the $\angle$ EDC $=$ the $\angle$ CDF, and the $\angle$ ECD $=$ the $\angle$ FCD

$\therefore$ the third $\angle$CED $=$ the third $\angle$CFD (Theor 16. Inf. 2)

In the $\triangle$ CFG, the ext. $\angle$ CFD is greater than the int. $\angle$ CGE (Theor. 8)

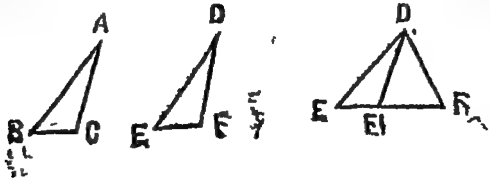
$\therefore$ the $\angle$ CEG is also greater than the $\angle$ CGE

$\therefore$ CG is greater than CE.

i. e., CE is less than CG. The oblique CE makes smaller angle with the perpendicular CD than the oblique CG makes with CD.

Q. E. D.

9. Let ABC and DEF two triangles in which AB $=$ DE, AC $=$ DF and the angle ABC $=$ the $\angle$DEF.



(*i*) It is required to prove that the ∠ ACB = the ∠ DFE and in this case the $\triangle^s$ ABC and DEF are equal in all respects

If the ∠ BAC be equal to the ∠ EDF.

Proof—Then in the two $\triangle^s$ ABC and DEF

because $\begin{cases} \text{BA} = \text{DE} \\ \text{AC} = \text{DF} \\ \text{and the } \angle \text{ BAC} = \text{the } \angle \text{ EDF} \end{cases}$

∴ the $\triangle^s$ are equal in all respects (Theor. 4)

so that, the ∠ ACB = the ∠ DFE.

It is required to prove that the ∠ ACB is the supplement of the ∠ DFE.

If the ∠ BAC be not equal to the ∠ EDF, let the ∠ EDF′ be equal to the ∠ BAC.

Proof—In the two $\triangle^s$ ABC and EDF′

Because $\begin{cases} \text{AB} = \text{ED} \\ \text{the } \angle \text{ ABC} = \text{the } \angle \text{ DEF}' \\ \text{and the } \angle \text{ BAC} = \text{the } \angle \text{ EDF}' \end{cases}$

∴ two $\triangle^s$ are equal in all respects (Theor 17)

So that AB = DF′ and the ∠ ACB = the ∠ DF′E

But AC = DF (given) ; ∴ DF = DF′

∴ the ∠ DFF′ = the ∠ DF′F (Theor. 5)

Now, the ∠ DF′F is supplement of the ∠ DF′E

∴ the ∠ DFF′ is supplement of the ∠ DF′E.

Or, the ∠ DFE is supplement of the ∠ DF′E

But the ∠ DF′E = the ∠ ACB

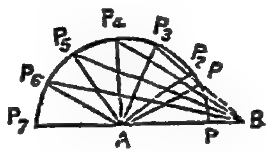
∴ the ∠ DFE is supplement of the ∠ ACB.

10. Let XY be a straight line of any length. From Q any point in XY draw PQ=4 cm perpendicular to XY. Through P draw obliques PA, PB, PC, PD, PE, making the angles

QPA, QPB, QPC QPD, QPE equal to 15°, 30°, 45°, 60°, 75° respectively with PQ.

Measure PA, PB, PC, PD, PE and it will be found that they are equal to 4·1 cm., 4·6 cm., 5 7 cm., 8 cm. and 15 6 cm respectively.

11. Let PAB be any triangle in which AP=3 cm. and AB=4 cm With centre A and radius =3 cm., draw a semi-circle $P_1P_4P_7$ As the angle A increases from 0° to 180°, the point P moves on the semi-circle from P_1 to P_7



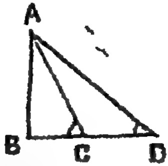
Let AP_2, AP_3, AP_4, AP_5, and AP_6 denote the successive positions of AP according as AP makes the $\angle P_2AB=30°$, $\angle P_3AB=60°$, $\angle P_4AB=90°$, $P_5AB=120°$, $P_6AB=150°$.

AP_1 denotes the position of AP when it makes an angle equal to 0° with AB, and AP_7 denotes the position of AP when it makes an angle equal to 180° with AB.

Join BP_2, BP_3, BP_4, BP_5, and BP_6.

Measure P_1B, P_2B, P_3B, P_4B, P_5B, P_6B and P_7B and it will be found that $P_1B=1$ cm., $P_2B=2$ cm, $P_3B=3·6$ cm $P_4B=4$ cm, $P_5B=6\ 1$ cm., $P_6B=6\ 8$ cm., and $P_7B=$ 7 cm.

12 Let C and D be any two points at a distance of 27″ apart. Join DC and produce it beyond C to any point B.



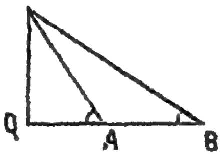
At C make the ∠ BCA = 65° and at D make the angle BDA = 40°, the arms CA and DA meeting in A. From A draw AB perpendicular to DC produced.

Then AB indicates the flagstaff.

Measure AB and it will be found 3·7″ long

∴ the flagstaff is 37 feet high.

13 Draw any vertical line PQ = 1·26″. From Q draw QB perpendicular to PQ. At P make the ∠ QPA = 33° and QPB = 57°, the arms PA, PB meeting QB in A and B respectively. Then the ∠ PAQ = 57° and the ∠ PBQ = 33° (Theor 16 Inf. 1).

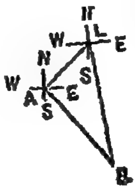


Then A and B indicate the two boats.

Measure AB and its will be found to be 1·12″

∴ The distance between two boats A and B is 112 ft.

14 From any point A draw a straight line AB=3″ in a south-easterly direction. At B make an angle ABL= 30° At A draw AL perpendicular to AB meeting BL in L



Then L indicates the lighthouse.

Measure AL and BL and it will be found that AL= 1·73″ and BL=3·465″.

∴ The distance of the lighthouse from A=346 yds, and its distance from B=693 yds

Page 59.

1. Let ABCD be a quadrilateral whose opposite sides are equal, *i e.*, AB=CD and AD=BC.

It is required to prove that the figure ABCD is a parallelogram

Join BD.

Proof —In the two $\triangle^s$ ABD and BCD

Because $\begin{cases} AB=CD \text{ (given)} \\ AD=BC \text{ (given)} \\ \text{and } BD \text{ is common to both} \end{cases}$

∴ two $\triangle^s$ are equal in all respects (Theor. 7)

so that, the ∠ ADB=the ∠ DBC and the ∠ ABD=the ∠ BDC.

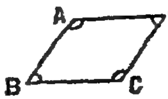
But these are alternate angles.

∴ AD is parallel to BC and AB is parallel to DC. (Theor. 13

∴ the figure ABCD is a parallelogram.

Q. E. D.

2 Let ABCD be a quadrilateral whose opposite angles are equal, *i e.*, the ∠ BAD=the ∠ BCD the ∠ ABC=the and ∠ ADC.



It is required to prove that the figure ABCD is a parallelogram.

Proof —The sum of the $\angle^s$ ABC, BCD, ADC and BAD of the quadrilateral ABCD is 4 rt $\angle^s$ (Theor. 16. Inf 5)

But the $\angle$ BAD=the $\angle$ BCD, and the $\angle$ ABC=the $\angle$ ADC (given)

$\therefore$ the $\angle$ BAD+ the $\angle$ ABC= 2 rt $\angle^s$

$\therefore$ AD and BC are parallel (Theor 13)

Again, because the $\angle$ ABC=the $\angle$ ADC (given) and the $\angle$ BAD+the $\angle$ ABC=2 rt. $\angle^s$ (proved)

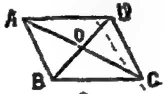
$\therefore$ the $\angle$ BAD+the $\angle$ ADC=2 rt $\angle^s$

$\therefore$ AB and CD are parallel (Theor. 13)

$\therefore$ the figure ABCD is a parallelogram.

Q E D.

3 Let the diagonals AC, BD of the quadrilateral ABCD bisect each other at O, ie, AO=OC, and OB=DO.



It is required to prove that the figure ABCD is a parallelogram.

Proof —In the two Δ^s AOD and BOC

Because $\begin{cases} \text{AO=CO (given)} \\ \text{DO=OB (given)} \\ \text{and the } \angle \text{ AOD=the } \angle \text{ BOC} \quad \text{(Theor 3)} \end{cases}$

$\therefore$ two Δ^s are equal in all respects (Theor 4)

so that, the $\angle$ ADO=the $\angle$ CBO and the $\angle$ DAO=the $\angle$ BCO

But these are alternate angles

$\therefore$ AD is parallel to BC, and AB parallel to CD (Theor. 13)

$\therefore$ the figure ABCD is a parallelogram.

Q. E. D.

4. (See Fig. in Ex. 10 on p. 26).

Let ABCD be a rhombus whose diagonals AC, BD cut one another at O.

It is required to prove that the diagonals AC, BD bisect one another at right angles.

Proof.—In the two $\triangle^s$ ADB and DCB

Because $\begin{cases} AD=BC \text{ (given)} \\ AB=DC \text{ (given)} \\ \text{and } BD \text{ is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 7)

so that, the $\angle$ ADB = the $\angle$ CBD.

In the two $\triangle^s$ ADO and CBO

Because $\begin{cases} \text{the } \angle ADO = \text{the } \angle CBO \text{ (proved)} \\ \text{the } \angle AOD = \text{the } \angle BOC \text{ (Theor. 3)} \\ \text{and } AD=BC \text{ (given)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, AO=OC and DO=OB

i. e the diagonals AC and BD of the rhombus ABCD bisect one another at O

Now, in the two $\triangle^s$ ADO and CDO

Because $\begin{cases} AD=DC \text{ (given)} \\ DO \text{ is common to both} \\ \text{and } AO=OC \text{ (proved)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 7)

so that the $\angle$ AOD = the $\angle$ DOC, but these are adjacent angles, therefore each is a right angle (From definition)

the $\angle$ AOD = the $\angle$ BOC (Theor. 3)
= 1 rt. $\angle$

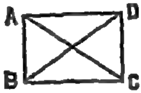
also the $\angle$ DOC = the $\angle$ AOB (Theor. 3)
= 1 rt $\angle$.

$\therefore$ the diagonals AC, BD of the rhombus ABCD cut one another at right angles at O

Hence the diagonals AD, BD of the rhombus ABCD bisect one another at right angles at O.

Q. E. D.

5. Let ABCD be a parallelogram whose diagonals AC, BD are equal.



It is required to prove that all its angles are right angles.

Proof.—In the two Δ^s ABC and ADB

Because $\begin{cases} \text{BC} = \text{AD} \quad \text{(Theor. 21)} \\ \text{AC} = \text{BD (given)} \\ \text{and AB is common to both} \end{cases}$

$\therefore$ two Δ^s are equal in all respects (Theor. 7)

so that, the $\angle$ ABC = the $\angle$DAB.

Because AD and BC are parallel and AB meets them

$\therefore$ $\angle$ ABC + $\angle$ DAB = 2 rt $\angle^s$ (Theor. 14).

But $\angle$ ABC = the $\angle$ DAB (proved)

$\therefore$ 2 $\angle$ ABC = 2 rt. $\angle^s$

or, $\angle$ ABC = 1 rt. $\angle$

also, 2 $\angle$ DAB = 2 rt. $\angle$

or, $\angle$ DAB = 1 rt. $\angle$

But the $\angle$ ABC = the $\angle$ ADC (Theor. 21).
= 1 rt. $\angle$
and the $\angle$ DAB = the $\angle$ DCB (Theor. 21)
= 1 rt. $\angle$

$\therefore$ all the angles of the parallelogram ABCD are right angles.

Q. E. D.

6. (See Fig in Ex. 3).

Let ABCD be a parallelogram which is not rectangular and let AC, BD be its diagonals

It is required to prove that DB and AC are not equal.

Proof.—Because the $\angle^s$ DAB and ABC = 2 rt. $\angle^s$ (Theor. 14).

and neither of the $\angle^s$ DAB and ABC is a rt. $\angle$

$\therefore$ one of them is acute and the other obtuse.

Let DAB be an acute angle, then ABC is an obtuse angle

Now, in the two $\triangle^s$ ADB and ABC

Because $\begin{cases} \text{AD = BC (Theor 21)} \\ \text{BA is common to both} \end{cases}$

but the $\angle$ DAB is less than the $\angle$ ABC,

$\therefore$ DB is less than AC. (Theor. 19)

$\therefore$ AC and DB are not equal

Q. E. D.

Page 60.

1. (See Fig. in Ex. 10 on p 26).

Let ABCD be a rhombus and let AC, BD be its diagonals.

It is required to prove that the rhombus ABCD is symmetrical about AC and BD.

Proof.—In the two triangles ABD and BCD

Because $\begin{cases} AB = BC \text{ (given)} \\ AD = DC \text{ (given)} \\ \text{and } DB \text{ is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 7)

so that, the $\angle$ ABD = the $\angle$ CBD and the $\angle$ ADB = the $\angle$ CDB.

i. e, BD bisects the $\angle^s$ ABC and ADC

$\therefore$ If the $\triangle$ ABD be turned about BD falling upon the $\triangle$ BCD then AD will fall upon DC and AB upon BC

But AD = DC and AB = BC

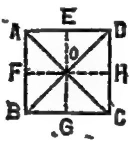
$\therefore$ the $\triangle$ ABD will coincide with the $\triangle$ BCD

$\therefore$ ABCD is symmetrical about BD.

Similarly it can be proved that the figure ABCD is symmetrical about AC.

Q E D

2. Let ABCD be a square whose diagonals AC, BD cut one another at O.



It is required to prove that the diagonals AC, BD of the square ABCD are axes of symmetry.

Proof.—In the two $\triangle^s$ ABD and BCD

Because $\begin{cases} AB = BC \text{ (given)} \\ AD = DC \text{ (given)} \\ \text{and } BD \text{ is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor 7)

so that, the $\angle$ ABD = the $\angle$ CBD and the $\angle$ ADB = the $\angle$ CDB

i. e., BD bisects the $\angle^s$ ABC and ADC.

$\therefore$ If the $\triangle$ ABD be turned about BD falling upon the $\triangle$ BCD, then AB will fall upon BC, and AD upon DC

But AB = BC and AD = DC

$\therefore$ the $\triangle$ ABD will coincide with the $\triangle$ BCD

$\therefore$ the figure ABCD is symmetrical about BD.

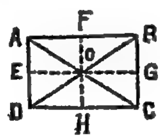
Similarly it can be proved that ABCD is symmetrical about AC

i. e., AC and BD are axes of symmetry of the square ABCD

Q. E. D.

A square is also symmetrical about its diameters, that is, the lines joining the middle points of its opposite sides, as shown by the dotted lines EG and FH in the figure.

3. Let ABCD be a rectangle whose diagonals AC, BD cut one another at O.



It is required to prove that the diagonals AC, BD divide the figure ABCD into two congruent triangles.

Proof—In the $\triangle^s$ ADB and DCB

Because $\begin{cases} AD = BC & \text{(Theor. 21)} \\ AB = DC & \text{(Theor. 21)} \\ \text{and } BD \text{ is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor 7)

so that, the $\angle$ ABD = the $\angle$ BDC and the $\angle$ ADB = the $\angle$ DBC.

Since two $\triangle^s$ ADB and DCB are congruent

$\therefore$ the diagonal DB divides the figure ABCD into two congruent triangles.

Similarly it can be proved that the diagonal AC divides ABCD into two congruent triangles.

Q. E. D.

In the $\triangle$ ABD, let AB be greater than AD then $\angle$ ADB is greater than $\angle$ ABD (Theor. 9)

But the $\angle$ ABD = the the $\angle$ BDC (proved)

$\therefore$ the $\angle$ ADB is greater than the $\angle$ BDC

Similarly the $\angle$ DBC is greater than $\angle$ ABD

i. e, the diagonal BD does not bisect the $\angle^s$ ABC and ADC

Similarly it can be proved that the diagonal AC does not bisect the $\angle^s$ DAB and DCA

$\therefore$ the diagonals AC and DB of the figure ABCD are not axes of symmetry, because the two parts will not coincide when the figure is folded about its diagonals.

A rectangle is symmetrical about its diameters, *i. e.*, the lines joining the middle points of its opposite sides, as shown by dotted lines EG and FH in the figure.

4. An oblique parallelogram has no axis of symmetry, because the diagonals do not bisect the angles through which they pass and diameters do not make equal angles with the sides whose middle points are joined by them

5. (See Fig. in Ex. 10 on p 49).

Let ABCD be a quadrilateral in which AB=AD and CB=CD, but the sides are not all equal.

It is required to find which of diagonals is an axis of symmetry.

Join AC and BD.

Proof —In the two $\triangle^s$ ABC and ADC

Because $\begin{cases} AB = AD \text{ (given)} \\ CB = CD \text{ (given)} \\ \text{and } AC \text{ is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 7)

so that, the $\angle$ BAC=the $\angle$ DAC and the $\angle$ ACB= the $\angle$ ACD i e, AC bisects the $\angle^s$ BAD and BCD.

$\therefore$ If the $\triangle$ ABC be turned about AC falling upon the $\triangle$ ADC, then AB will upon AD and CB upon CD.

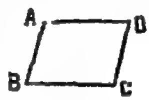
But AB=AD and CB=CD (given)

$\therefore$ the $\triangle$ ABC will coincide with the $\triangle$ ADC.

The diagonal BD does not bisect the $\angle^s$ ABC and ADC.

Hence, the figure ABCD is not symmetrical about DB.

6. (i) Let ABCD and EFGH be two parallelograms having two adjacent sides AB, BC of one respectively equal to two adjacent sides EF, FG of the other, each to each, and the angle BAD of the former equal to the angle FEH of the latter



It is required to prove that the parallelograms ABCD and EFGH are identically equal.

Proof.—Because the $\angle^s$ BAD and ABC=2 rt $\angle^s$ also the $\angle^s$ FEH and EFG= 2 rt $\angle^s$ (Theor 14)

$\therefore$ the $\angle^s$ BAD and ABC=the $\angle^s$ FEH and EFG

But the $\angle$ BAD=the $\angle$ FEH (given)

$\therefore$ the $\angle$ ABC=the $\angle$ EFG

Similarly it can be proved that the $\angle$ BCD=the $\angle$ FGH

Again because AB=CD and EF=GH (Theor 21)

and AB=EF (given)

$\therefore$ CD=GH

Similarly it can be proved that AD=EH.

Apply the parallelogram ABCD to the parallelogram EFGH.

so that, the point B falls on F and the sides BC along FG,

But BC=FG (given)

∴ the point C will falls on G

And because the ∠ ABC=the ∠ EFG (proved)

∴ the side BA will fall along FE

But AB=EF (given)

∴ the point A will fall on E

Again because the ∠ BCD=the ∠ FGH (proved)

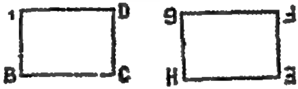
∴ the side CD will fall along GH

But CD=GH (proved)

∴ the point D falls on the point H, and hence the side AD falls on the side EH

∴ the parallelogram ABCD will coincide with the parallelgram EFGH and will therefore be identically equal to it.

(ii) Let ABCD and EFGH be two rectangles having two adjacent sides AB, BC of the one respectively equal to two adjacent sides EF, FG of the other, each to each



It is required to prove that the rectangles ABCD and EFGH are identically equal.

Proof — All the angles of rectangles are right angles (Theor. 21. Cor. 1)

Because AB=CD and EF=GH (Theor. 21)

and AB=EF (given)

∴ CD=GH

Similarly it can be proved that AD=EH

Apply the rectangle ABCD to the rectangle EFGH so that the point B falls on F and BA along FE, then because the ∠ ABC=the ∠ EFG (being rt. ∠s), therefore BC will fall along FG.

Now because AB=EF and BC=FG (given)

∴ the point A will fall on E and C on G.

Again because the ∠ BAD=the ∠ FEH (being rt. ∠s)

and the ∠ BCD=the ∠ FGH (being rt ∠s)

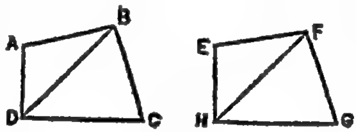
∴ the side AD will fall along EH and CD along GH

∴ the point D will fall on the point H.

∴ the rectangle ABCD coincides with the rectangle EFGH, and is therefore identically equal to it.

Q. E. D.

7. Let ABCD and EFGH be two quadrilaterals in which AB=EF, BC=FG, CD=GH and DA=HE, also the ∠ BAD = the ∠ FEH.



It is required to prove that the figures ABCD and EFGH may be made to coincide with one another.

Proof—Apply the figure ABCD to the figure EFGH, so that A falls on E and AD along EH, then because the ∠ BAD=the ∠ FEH, therefore AB will fall along EF.

Now because AB=EF and AD=EH (given)

∴ the point B will fall on F and the point D on H.

∴ the straight line BD must coincide with the straight line HF, for otherwise two straight lines would enclose a space.

Now BD coinciding with HF, and the sides DC, BC being respectively equal to the sides, HG, FG, each to each,

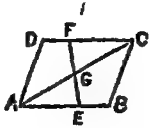
∴ the △ DBC will coincide with the △ HFG

(Theor. 7)

∴ the whole figure ABCD will coincide with the whole figure EFGH.

Q. E. D.

8. Let ABCD be a parallelogram and let AC be a diagonal. Let EF be a straight line passing through G, the middle point of AC and terminated by a pair of opposite sides AB, DC at E and F.



It is required to prove that EF is bisected at G.

Proof—In the two $\triangle^s$ AEG and FGC

Because $\begin{cases} AG = GC \text{ (given)} \\ \text{the } \angle EAG = \text{the alternate } \angle GCF \\ \text{and the } \angle AGE = \text{ the } \angle FGC \quad \text{(Theor. 3)} \end{cases}$

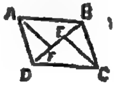
$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, EG = GF

i e, EF is bisected at G

Q E. D

9. Let ABCD be a parallelogram and let BD be any diagonal Let AF and CE be perpendiculars drawn from A and C to BD.



It is required to prove that AF = CE

Proof.—In the two $\triangle^s$ AFD and BEC

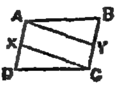
Because $\begin{cases} \text{the } \angle AFD = \text{the } \angle CEB \text{ (being rt } \angle^s) \\ \text{the } \angle ADF = \text{the alternate } \angle EBC \\ \text{and } AD = BC \quad \text{(Theor 21)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, AF = CE

Q. E D

10. Let ABCD be a parallelogram and let X, Y respectively be the middle points of the sides AD, BC. Join CX and AY.



It is required to prove that the figure AYCX is a parallelogram.

Proof.—AD is equal and parallel to BC (Theor. 21)

But AX = ½ AD and CY = ½ BC (given)

∴ AX is equal and parallel to CY

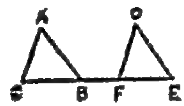
The extremities of two equal and parallel straight lines AX and CY are joined towards the same parts by the straight lines AY and CX

∴ AY is equal and parallel to CX (Theor. 20)

∴ the figure AYCX is a parallelogram

Q. E. D.

11 Let ABC and DEF be two triangles such that AB, BC are respectively equal and parallel to DE, EF.



It is required to prove that AC is equal and parallel to DF.

Place the triangles such that their bases are in the same straight line.

Then in the two $\triangle^s$ ABC and DEF

Because $\begin{cases} AB = DE \text{ (given)} \\ BC = EF \text{ (given)} \\ \text{and the ext } \angle\ ABC = \text{the int. } \angle\ DEF \text{ (Theor. 14)} \end{cases}$

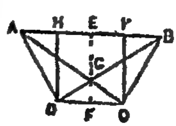
∴ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, AC = DF, and the int ∠ ACB = the ext. ∠ DFE,

∴ AC is parallel to DF (Theor. 13), and also equal to it.

Q. E. D.

12. Let ABCD be a quadrilateral in which AD is equal but not parallel to BC, and AB is parallel to DC,



(2) It is required to prove that the $\angle$ BAD + the $\angle$ BCD $= 180°$ = the $\angle$ ABC + the $\angle$ ADC

From D and C draw DH and CK perpendiculars to AB meeting AB in H and K.

Proof—In the figure DHKC, the two int. $\angle^s$ DHK and HKC = 2 rt $\angle^s$

$\therefore$ DH is parallel to KC (Theor. 13)

But HK is parallel to DC (given) and the $\angle^s$ DHK and HKC are rt $\angle^s$ (by construction)

$\therefore$ the figure DHKC is a rectangle.

$\therefore$ DH = KC (Theor 21)

In the two $\triangle^s$ ADH and BKC

because $\begin{cases} AD = BC \text{ (given)} \\ DH = KC \text{ (proved)} \\ \text{and the } \angle AHD = \text{the } \angle BKC \text{ (being rt. } \angle^s) \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 18)

so that, the $\angle$ DAH = the $\angle$ KBC and AH = BK

Since AB and DC are parallel and AD meets them

$\therefore$ the $\angle^s$ BAD + ADC = 2 rt $\angle^s = 180°$ (Theor 14)

But the $\angle$ BAD = the $\angle$ ABC (proved)

$\therefore$ the $\angle^s$ ABC + ADC $= 180°$

Again, because AB and DC are parallel and BC meets them

$\therefore$ $\angle^s$ ABC + BCD = 2rt. $\angle^s = 180°$ (Theor 14)

But $\angle$ ABC = $\angle$ BAD

$\therefore$ the $\angle^s$ BAD + BCD $= 180°$

$\therefore$ the $\angle$ BAD + the $\angle$ BCD $= 180°$ = the $\angle$ ABC + the $\angle$ ADC

(*ii*) Join AC and BD.

It is required to prove that the diagonal AC=the diagonal BD.

Proof—In the two $\triangle^s$ ADB and ABC

because $\begin{cases} \text{AD=BC (given)} \\ \text{AB is common to both} \\ \text{and the } \angle \text{ DAB=the } \angle \text{ ABC (proved in (}i\text{))} \end{cases}$

$\therefore$ the $\triangle^s$ are equal in all respects (Theor. 4)

so that, BD=AC

(*iii*) Let E and F be the middle points of AB and DC respectively. Join EF.

It is required to prove that the figure ABCD is symmetrical about EF.

The figure HKCD is a rectangle (proved in (*i*))

and C,F are the middle points of HK, CD ($\because$ AH=BK and AE=BE)

$\therefore$ EF is perpendicular to HK and DC.

$\therefore$ If the figure ABCD be turned about EF, so that the figure ADFE falls upon the figure BCFE, then because the $\angle$ AEF=the $\angle$ BEF and the $\angle$ EFD=the $\angle$ EFC (being proved to be rt. $\angle^s$), therefore AE will fall along EB and DF along FC

But AE=EB and DF=FC (given)

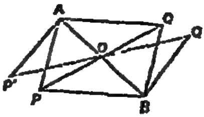
$\therefore$ the point A will fall on B and D on C.

$\therefore$ AD will fall along BC, and coincide with it.

$\therefore$ the figure ABCD is symmetrical about EF.

Q. E. D.

13. Let AP and BQ indicate the positions of the equal straight rods at the time of starting, being parallel but pointing in opposite senses, and AP′ and BQ′ their positions at the end of any interval of time.



Join AQ, PB, AB and PQ, and let PQ, AB cut at O

(i) It is required to prove that AP′ is parallel to BQ′

Since the lines AP and BQ are initially parallel,

∴ ∠ PAB = the alternate ∠ QBA (Theor. 14)

Then since the rods AP and BQ are turning at equal rates in clockwise direction about A and B, the angles through which they have turned, *viz* ∠s PAP′ and QBQ′ are equal.

By adding we have

the ∠s PAP′ + PAB = the ∠ QBQ′ + ABQ

or, the ∠ P′AB = the ∠ Q′BA, but these are alternate angles

∴ AP′ and BQ′ are parallel (Theor. 13)

Thus the lines will always be parallel.

(ii) It is required to prove that the line joining P and Q will always pass through a certain fixed point.

Proof—Because AP is equal and parallel to BP

∴ the straight lines AQ and BP are equal and parallel (Theor 20)

∴ the figure AQBP is a parallelogram.

∴ its diagonals AB and PQ bisect each other at O. (Theor 21 Cor 3)

Again, because A, B are fixed points, therefore O, the middle point of AB, is also a fixed point.

$\therefore$ PQ passes through a fixed point O.

Q. E. D.

14. It is required to calculate the angles of a $\triangle$ ABC if the int. $\angle$ A$=\frac{3}{7}$ of ext. $\angle$ A, 3B$=$4C.

Because the int. $\angle$ A$+$ext. $\angle$ A$=180°$, and the int $\angle$ A$=\frac{3}{7}$ of ext. $\angle$ A

$\therefore$ $\frac{3}{7}$ of ext. $\angle$ A$+$ext. $\angle$ A$=180°$

or, $\frac{10}{7}$ of ext $\angle$ A$=180°$

$\therefore$ ext. $\angle$ A$=180° \times \frac{7}{10}=126°$

$\therefore$ int. $\angle$ A$=180°-126°=54°$

In the $\triangle$ ABC, $\angle A + \angle B + \angle C = 180°$ (Theor 16 Inf. 1)

and the $\angle A = 54°$

$\therefore \angle B + \angle C = 180° - 54° = 126°$

$\therefore \frac{4}{3} \angle C + \angle C = 126°$

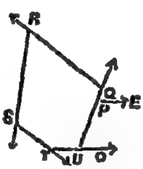
or, $\frac{7}{3} \cdot \angle C = 126°$

$\therefore \angle C = 126° \times \frac{3}{7} = 54°$

$\angle B + \angle C = 126°$ and $\angle C = 54°$

$\therefore \angle B = 126° - 54° = 72°$.

15. Let P, Q, R, S, represent the points at which the yacht changes her course successively by 63°, by 78°, by 119° and by 64°, starting from the point P and sailing due east Let T O be the direction to which she had been set so that she is again moving in an easterly direction. QP is produced te meet TO at U,



It is required to find what change have been made to set the yacht once more on an easterly course.

Because T O is parallel to P E

$\therefore$ the $\angle$ PUO$=$the $\angle$ QPE$=63°$.

Now QRSTU forms a pentagon

$\therefore$ the sum of all its ext angles $= 4$ rt. $\angle^s = 360^\circ$ (Theor. 16 Cor 2)

The sum of the ext. angles at U, Q, R, S$=63^\circ+78^\circ+119^\circ+64^\circ$

$\therefore$ the ext angle at T$=360^\circ-324^\circ=36^\circ$

Thus the yacht must change her course by 36°.

16 The sum of all the ext angles of a rectilineal figure $= 4$ rt $\angle^s$ (Theor. 16 Cor 2)

$\therefore$ the sum of the int. angles of the figure $= 4$ rt $\angle^s$ (given)

$\therefore$ the sum of the ext and int. angles of the figure $= 4$ rt $\angle^s + 4$ rt. $\angle^s$, or, 8 rt $\angle^s$

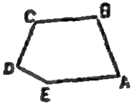
But the ext and int angles at any vertex $= 2$ rt $\angle^s$

$\therefore$ the number of vertices and hence of sides $= \dfrac{8 \text{ rt } \angle^s}{2 \text{ rt } \angle^s} = 4$

$\therefore$ the figure contains 4 sides.

$\therefore$ the given figure is a quadrilateral

17. It is required to construct a pentagon or five-sided figure ABCDE having given that the $\angle$ B$=110^\circ$ the $\angle$C$=$ the 115°, $\angle$D$=93^\circ$ and the $\angle$ E$=152^\circ$.



Take a straight line EA, of any convenient length. At E make the $\angle$ AED$=152^\circ$ At D make the $\angle$ EDC$=93^\circ$.

At C make the $\angle$ DCB$=115^\circ$. At B make the $\angle$ CBA $=110^\circ$ the arm BA meeting EA in A.

Then ABCDE is the required five-sided figure

It is required to prove that AE is parallel to BC

The sum of the int. angles of the pentagon ABCDE

$= 2.5$ rt $\angle^s - 4$ rt $\angle^s$ (Theor. 16. Cor. 1)

$= 10$ rt $\angle^s - 4$ rt. $\angle^s = 6$ rt $\angle^s = 540°$

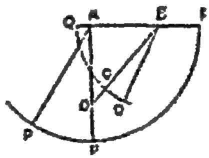
the sum of the $\angle^s$ B, C, D and E $= 110° + 115° + 93° + 152°$ $= 470°$

$\therefore \angle A = 540° - 470° = 70°$

$\therefore \angle EAB + \angle ABC = 70° + 110° = 180° = 2$ rt $\angle^s$

$\therefore$ AE is parallel to BC (Theor. 13)

18. Let AP, AQ indicate the positions of the rods at the time of starting; AP starting from the direction AB and turning clockwise at the uniform rate of $7\frac{1}{2}°$ a second about A, and BQ starting simultaneously from the direction BA and turning counter clockwise at the rate of $3\frac{3}{4}°$ a second about B.



(i) It is required to find the time that will elapse before AP and BQ are parallel.

Let AP′, BQ denote their positions when they are parallel. Then the sum of the $\angle^s$ P′AP and QBQ′ together equal to 2 rt angles, or 180° (Theor. 14)

and because the sum of the angles through which they turn in one second is $7\frac{1}{2}° + 3\frac{3}{4}°$, or, $11\frac{1}{4}°$

$\therefore$ They will be parallel after $\dfrac{180°}{11\frac{1}{4}°}$, or, 16 seconds after the start.

(ii) It is required to calculate the angle between AP and BQ twelve seconds from the start.

Let AP″, BP″ denote their positions 12 seconds after the start.

Then the $\angle$ PAP$''$ $=12\times7\frac{1}{2}^\circ$ or 90° and the $\angle$ QBQ$'$

$=12\times3\frac{3}{4}^\circ$, or, 45°

$\therefore$ the angle between AP and BQ twelve seconds from the start$=90^\circ-45^\circ$, or 45°.

(iii) It is required to find the rate at which the angle between AP and BQ decreases

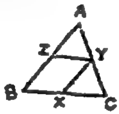
Because AP.BQ turn through $7\frac{1}{2}^\circ+3\frac{3}{4}^\circ$, or $11\frac{1}{4}^\circ$ in a second.

And because the angle between AP and BQ diminishes each second by the amount by which the sum of the $\angle^s$ BAP QBA is increased, for the sum of the three angles is constant

Hence, the rate of decrease is $11\frac{1}{4}^\circ$ in a second.

Page 64

1. Let ABC be a triangle and let Z be the middle point of AB from which ZY is drawn parallel to BC meeting AC in Y



It is required to prove that ZY bisects the side AC i e, AY$=$YC.

From Y draw YX parallel to AB meeting BC in X

Proof —Because ZY is parallel to BX and YX parallel to ZB, therefore the figure ZYXB is a parallelogram.

$\therefore$ the side ZB$=$YX (Theor. 21)

But ZB$=$AZ (given), $\therefore$ AZ$=$YX

Because AB and XY are parallel and AC meets them

$\therefore$ the $\angle$ BAC or the $\angle$ ZAY$=$the $\angle$ XYC (Theor. 14)

Again, because ZY and BC are parallel and AC meets them

$\therefore$ the $\angle$ AYZ = the $\angle$ ACB or the $\angle$ YCX (Theor. 14)

Now, in the two $\triangle^s$ AZY and YXC

because $\begin{cases} \text{AZ} = \text{YX (proved)} \\ \text{the } \angle \text{ AYZ} = \text{the } \angle \text{ YCX (proved)} \\ \text{and the } \angle \text{ ZAY} = \text{the } \angle \text{ XYC (proved)} \end{cases}$

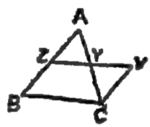
$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, AY = YC

i. e., AC is bisected at Y.

Q. E. D.

2. Let ABC be a triangle, and let Z and Y be the middle points of AB and AC. Join ZY.



It is required to prove that ZY is parallel to BC.

Produce ZY to any point V making YZ = VY. Join VC.

Proof —In the two $\triangle^s$ AZY and YVC.

because $\begin{cases} \text{AY} = \text{YC (given)} \\ \text{ZY} = \text{YV (by construction)} \\ \text{and the } \angle \text{ AYZ} = \text{the } \angle \text{ VYC (Theor. 3)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, AZ = VC and the $\angle$ ZAY = the $\angle$ YCV, but the $\angle^s$ ZAY and YCV are alternate angles

$\therefore$ AZ or AB is parallel to VC (Theor. 13)

but AZ = ZB (given); $\therefore$ ZB = VC.

Two equal and parallel straight lines ZB and VC are joined towards the same parts by ZV and BC,

$\therefore$ ZV and BC are parallel (Theor. 20)

or, ZY is parallel to BC.

Q. E. D.

3. (See figure in Ex. 2).

Let ABC be a triangle, and Z, Y the middle points of AB, AC respectively. Join ZY.

It is required to prove that ZY is half of BC.

Produce ZY to any point V making YV=ZY. Join VC.

Proof—In the two $\triangle^s$ AZY and YVC

Because $\begin{cases} AY = YC \text{ (given)} \\ ZY = YV \text{ (by construction)} \\ \text{and the } \angle\, AYZ = \text{the } \angle\, VYC \quad \text{(Theor. 3)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, AZ=VC and the $\angle$ AZY=the $\angle$ YVC

but the $\angle^s$ AZY and YVC are alternate angles

$\therefore$ AZ or AB is parallel to VC (Theor. 13)

also AZ=ZB (given), $\therefore$ ZB=VC

Two equal and parallel straight lines ZB and VC are joined towards the same parts by ZV and BC,

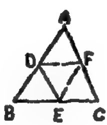
$\therefore$ ZV and BC are equal and parallel (Theor. 20)

But ZY=YV (by construction)=$\frac{1}{2}$ ZV

$\therefore$ ZY=$\frac{1}{2}$ BC.

Q. E. D.

4. Let ABC be a triangle and D, E, F, the middle points of AB, BC, CA respectively. Join DE, EF and FD.



It is required to prove that, the straight lines DE, EF, FD divide the $\triangle$ ABC into four triangles ADF, DBE, DEF and FEC which are identically equal.

Proof—Because D is the middle point of AB, and F is the middle point of AC

$\therefore$ DF = $\frac{1}{2}$ BC (proved in Ex. 3)

Similarly, it can be proved that DE = ½ AC, and FE = ½ AB

But BE = EC (given) = ½ BC

∴ DF = BE = EC

Similarly, DE = AF = FC, and EF=AD=DB.

Now, in the two $\triangle^s$ ADF and DBE

$$\text{Because}\begin{cases}\text{AD=DB (given)}\\ \text{AF=DE (moved)}\\ \text{and DF=BE (proved)}\end{cases}$$

∴ two $\triangle^s$ are equal in all respects (Theor. 7)

Again, in the two $\triangle^s$ ADF and FEC

$$\text{Because}\begin{cases}\text{AF=FC (given)}\\ \text{AD=FE (proved)}\\ \text{and DF=EC (proved)}\end{cases}$$

∴ two $\triangle^s$ are equal in all respects (Theor. 7)

Again, in the two $\triangle^s$ ADF and DEF

$$\text{Because}\begin{cases}\text{AD=EF (proved)}\\ \text{AF=DE (proved)}\\ \text{and DF is common to both}\end{cases}$$

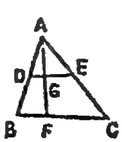
∴ two $\triangle^s$ are equal in all respects (Theor 7)

∴ four triangles ADF, BDE, FEC and DEF are identically equal.

i e, the △ ABC in divided into four triangles ADF, BDE, DEF and FEC by DE, EF, FC and these four triangles are identically equal.

Q. E. D.

5. Let ABC be a triangle and D, E be the middle points of AB, AC respectively. Join DE. Let AF be any straight line drawn from the vertex A to the base BC cutting DE in G.



It is required to prove that AF is bisected at G by DE

Proof —Because D is the middle point of AB, and E the middle point of AC

∴ DE is parallel to BC (proved in Ex. 2)

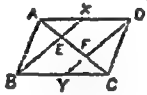
In the △ ABF, because D is the middle point of AB, and DG is parallel to BF

∴ DG bisects AF at G

What is true of AF is also true of any other straight line drawn from the vertex A to the base BC

Q. E D.

6. Let ABCD be a parallelogram, and X, Y the middle points of the opposite sides AD BC.



Join BX, DY Join AC cutting BX and DY at E and F respectively

It is required to prove that the diagonal AC is trisected by BX and DY, or AE=EF=FC

Proof —In the parallelogram ABCD, AD=BC (Theor. 21)

and XD=½AD, BY=½BC given)

∴ XD=BY, also XD and BY are parallel.

Two equal and parallel straight lines XD and BY are joined towards the same parts by BX and DY

∴ BX and DY are parallel (Theor. 20)

Now, in the △ AFD, because X is the middle point of AD, and XE is parallel to FD proved)

∴ XE bisects AF, or AE=EF (proved in Ex. 1)

Similarly, in the △ BEC, because Y is the middle point of BC, and YF is parallel to BE (proved)

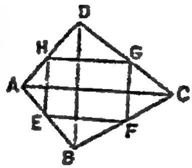
∴ YF bisects EC, or EF=FC (proved in Ex. 1)

∴ AE=EF=FC.

i. e., the diagonal AC is trisected by BX and DY

Q. E. D.

7. Let ABCD be a quadrilateral and E, F, G, H be the middle points of AB, BC, CD, and DA respectively. Join EF, FG, GH and HE.



It is required to prove that the figure EFGH is a parallelogram.

Join AC and BD.

Proof—In the △ ABC, because E and F are the middle points of AB, BC respectively.

∴ EF is parallel to AC (proved in Ex. 2)

Similarly, in the △ ADC, H, G being the middle points of AD, DC respectively, HG is parallel to AC

∴ EF is parallel to HG (Theor 15)

Again, in the △ ABD, because E and H are the middle points of AB, AD respectively

∴ EH is parallel to BD (proved in Ex. 2)

Similarly, in the △ BDC, F, G being the middle points of BC, CD respectively, FG is parallel to BD

∴ EH is parallel to FG (Theor 15)

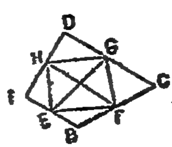
∴ in the quadrilateral EFGH, because EF is parallel to HG, and EH parallel to FG.

∴ the quadrilateral EFGH is a parallelogram

(From definition)

Q. E. D.

8. Let ABCD be a quadrilateral and E, F, G, H, be the middle points of AD, BC, CD, DA respectively.



Join EG and FH cutting at O.

It is required to prove that EG and FH bisect one another at O.

Join EF, FG, GH and HE

Proof—In the quadrilateral ABCD, because E,F,G,H are the middle points of AB,BC,CD,DA respectively

∴ the figure EFGH is a parallelogram (proved in Ex. 7)

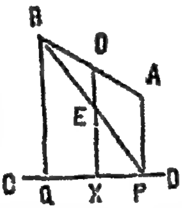
Now, the diagonals of a parallelogram bisect one another (Theor. 21 Cor 5)

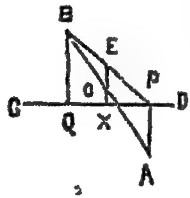
∴ the diagonals EG, FH of the parallelogram EFGH bisect one another at O.

i. e., the straight lines EG,FH which join the middle points of opposite sides of the quadrilateral ABCD bisect one another.

Q E D.

9. Let CD be a straight line and AB another straight line whose middle point is O Let BQ, OX, AP be perpendiculars from B,O and A to CD





It is required to prove that $OX = \frac{1}{2}(AP + BQ)$, or $\frac{1}{2}(AP - BQ)$, according as A and B are on the same side, or on opposite sides of CD.

When A and B are on the same side of CD.

Join BP cutting OX at E.

Proof—Because BQ,OX and AP are perpendiculars to CD

∴ BQ,OX and AP are parallel to one another (proved in Ex. 2 on page 41)

In the △ ABP, because O is the middle point of AB, and OE is parallel to AP

∴ E is the middle point of BP (proved in Ex. 1)

$\therefore$ OE + EX$=\frac{1}{2}$(AP + BQ) (by adding together)

or, OX$=\frac{1}{2}$ (AP+BQ)

When A and B are on opposite sides of CD, AB and CD cut one another.

Join BP and produce XO to meet BP in E.

Because BQ, OX and AP are perpendiculars to CD

$\therefore$ BQ, OX and AP are parallel to one another (proved in Ex. 2 on page 41)

In the $\triangle$ BAP because O is the middle point of AB and OE is parallel to AP

$\therefore$ E is the middle point BP (proved in Ex. 1)

Again, in the $\triangle$ BAP because O and E are the middle points of AB and BP respectively,

$\therefore$ OE is $\frac{1}{2}$ AP (proved in Ex. 3)

In the $\triangle$ BQP, because E is the middle point of BP (proved), and EX is parallel to BQ

$\therefore$ X is the middle point of QP (proved in Ex. 1)

Again, in the $\triangle$ BQP, because E, X are the middle points of BP, QP respectively

$\therefore$ EX is $\frac{1}{2}$ of BQ (proved in Ex. 3)

Again, because OE is $\frac{1}{2}$ of AP and EX $\frac{1}{2}$ of BQ.

$\therefore$ EX$-$EO$=\frac{1}{2}$ (BQ$-$AP)

or, OX$=\frac{1}{2}$ (BQ$-$AP)

If AP is greater than BQ, then the middle point O of AB will be on the same side as the point A. In this case, OX $=\frac{1}{2}$ (AP$-$BQ).

Q. E. D.

If $AP = 4{\cdot}2$ cm. and $BQ = 5{\cdot}8$ cm.

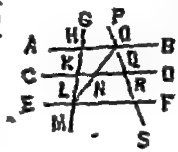
then $OX = \frac{1}{2}(AP + BQ) = \frac{1}{2}(4{\cdot}2 + 5{\cdot}8)$ cm.

$= \frac{1}{2}$ of 10 cm. $= 5$ cm.

or, $OX = \frac{1}{2}(BQ - AP) = \frac{1}{2}(5{\cdot}8 - 4{\cdot}2)$ cm.

$= \frac{1}{2}$ of $1{\cdot}6$ cm $= {\cdot}8$ cm.

10. Let AB, CD, EF be three parallel straight lines which cut off equal intercepts from two transversals GM and PS, so that $HK = KL$, and $OQ = QR$.



It is required to prove that KQ is the ARITHMETIC MEAN of HO and LR.

Join OL cutting KQ at N.

Proof.—In the $\triangle$ HLO, K is the middle point of HL (given), and KN is parallel to HO,

$\therefore$ N is the middle point of LO (proved in Ex. 1)

$\therefore$ KN is $\frac{1}{2}$ of HO (proved in Ex. 3)

Again, in the $\triangle$ OLR, N is the middle point of LO (proved), and Q is the middle point of OR (given).

$\therefore$ NQ is $\frac{1}{2}$ of LR (proved in Ex. 3)

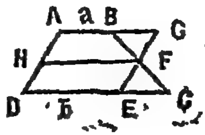
$\therefore KN + NQ = \frac{1}{2}(HO + LR)$

or, $KQ = \frac{1}{2}(HO + LR)$

i. e. KQ is the ARITHMETIC MEAN of HO and LR.

Q. E. D.

11. Let ABCD be a trapezium in which AB, DC are parallel, and equal to a cm. and b cm. respectively, and H, F the middle points of AD, BC respectively.



Join HF.

It is required to prove that HF is parallel to AB or CD.

Through F draw the straight line EFG parallel to AD meeting DC in E. Produce AB beyond B to meet EFG in G.

Proof.—In the quadrilateral ADEG, because AG is parallel to DE, and AD parallel to GE

∴ the quadrilateral ADEG is a parallelogram

∴ AD=EG and AG=DE (Theor. 21)

Because BG and EC are parallel, and GE meets them

∴ the ∠ BGE or the ∠ BGF=alternate ∠ GEC or the ∠ FEC (Theor. 14)

Now, in the two $\triangle^s$ BFG and FEC

Because $\begin{cases} \text{BF=FC (given)} \\ \text{the } \angle \text{ BGF=the } \angle \text{ FEC (proved)} \\ \text{and the } \angle \text{ BFG=the } \angle \text{ EFC (Theor. 3)} \end{cases}$

∴ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, FG=EF and BG=EC.

Because AH=$\frac{1}{2}$ AD (given); FG=EF=$\frac{1}{2}$ GE, and AD=EG (proved)

∴ AH=GF, and since they are parallel

∴ HF and AG are also equal and parallel (Theor. 20)

i.e HF is parallel to AB

$\therefore$ HF is also parallel to DC (Theor. 15)

Now, HF=AG=DE (proved)

$\therefore$ HF$=\frac{1}{2}$ (AG+DE)

But AG=AB+BG, and DE=DC−EC

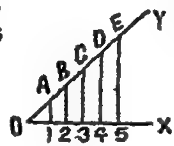
$\therefore$ AG+DE=AB+DC (since BG=EC, proved)

$\therefore$ HF$=\frac{1}{2}$ (AB+DC)

$=\frac{1}{2}(a+b)$ cm.

Q. E. D.

12 Let OX and OY be any two straight lines meeting at O and at any angle. Along OX five points, 1, 3, 4, 5 are marked at equal distances.



Take any point E in OY.

Join E 5.

From 4, 3 2, 1 draw parallels D4, C3, B2, A1 to E5 meeting OY in D, C, B, A.

It is required to prove that C3 is the mean of all five parallels A1, B2, C3, D4, E5.

Proof.—Because the straight line OX is divided into equal parts by 1, 2, 3, 4, 5 , and from 1, 2, 3, 4, 5, parallel straight lines are drawn cutting OY in A, B, C, D, E.

$\therefore$ OE is also divided into equal parts by A, B, C, D, E

(Theor 22. Cor.)

Again, because BD42 is a trapezium, and C, 3 are the middle points of BD, 24 respectively

$\therefore$ C3$=\frac{1}{2}$ (B2+D4) (proved in Ex. 11)

$\therefore$ 2.C3=(B2+D4)

Again, because 1AE5 is a trapezium, and C, 3 are the middle points of AE, 15 respectively

$\therefore$ C3 $= \frac{1}{2}$ (A1 + E5) (proved in Ex. 11)

$\therefore$ 2 C3 $=$ (A1 + E5)

$\therefore$ A1 + B2 + C3 + D4 + E5 $=$ 2.C3 + 2 C3 + C3

$= 5$.C3

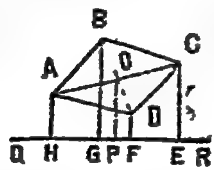
$\therefore$ C3 $= \frac{1}{5}$ (A1 + B2 + C3 + D4 + E5)

i.e. C3 is the MEAN of all the five parallels.

Q. E. D.

The corresponding theorem for any odd number $(2n+1)$ of parallels so drawn is that the $(n+1)$ parallel is the mean of all the $(2n-1)$ parallels.

13. Let ABCD be a parallelogram and QR any straight line outside it. From A, B, C, D perpendiculars AH, BG, DF, CE are drawn to QR.



It is required to prove that the sum of the perpendiculars AH and CE is equal to the sum of the perpendiculars BG and DF.

Join AC and BD cutting at O. From O draw OP perpendicular to QR.

Proof.—The diagonals AC, BD of the parallelogram ABCD bisect one another at O (Theor. 22. Cor. 3)

Because O is the middle point of AC, and from A, O, C perpendiculars AH, OP, CE are drawn to QR

$\therefore$ OP $= \frac{1}{2}$ (AH + CE) (proved in Ex. 9)

Again, because O is the middle point of BD, and from B, O, D perpendiculars BG, OP, DF are drawn to QR

$\therefore$ OP $= \frac{1}{2}$ (BG + DF) (proved in Ex. 9)

$\therefore$ $\frac{1}{2}$ (AH + CE) $= \frac{1}{2}$ (BG + DF)

or, AH+CE = BG+DF

Q. E. D.

14. Let ABC be an isosceles triangle in which AB, AC are equal. Let D be any point in BC, let DE, DF be perpendicular from D to AB, BC respectively, and CG be perpendicular from C to AB.

It is required to prove that CG is equal to the sum of DE and DF.

From D draw DH perpendicular to CG

Proof.—

The ∠EGH is a rt. ∠, and the ∠GHD is also a rt. ∠ (by construction)

∴ the ∠s EGH + GHD=2 rt. ∠s

∴ EG and HD are parallel (Theor. 14)

Similarly GH and DE are parallel

In the quadrilateral EDHG, GH is parallel to ED and EG parallel to HD

∴ the quadrilateral EDHG is a parallelogram

so that, ED=GH (Theor. 21)

In the △ ABC, because AB=AC (given)

∴ the ∠ABC = the ∠ACB (Theor. 5)

Now since AB and HD are parallel, and BC meets them

∴ the ∠ABC = the ∠HDC (Theor. 13)

∴ the ∠HDC = the ∠ACB or the ∠FCD.

Now, in the two △s HDC and FDC

Because { the ∠DHC= the ∠DFC (being right angles)
the ∠HDC= the ∠FCD (proved)
and DC is common to both

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, $CH = DF$

Also, $GH = DE$ (proved)

$\therefore CH + GH = DF + DE$

or, $CG = DF + DE$

Similarly, by drawing a perpendicular from B to AC it can be proved that the perpendicular is equal to the sum of DF and DE.

It can be proved by drawing perpendiculars from any point in BC to the equal sides AB, AC, that the sum of these perpendiculars is equal to the perpendicular drawn either from C on AB or from B on AC.

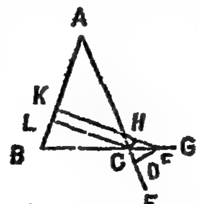
But the perpendicular drawn either from C on AB or from B on AC is constant.

$\therefore$ the sum of the perpendiculars drawn from any point in BC to the equal sides is also constant.

Q. E. D.

It is required to find the modification in the above proved property, when the perpendiculars are drawn from any point in the prolongation of the base BC to the equal sides AB, AC.

Let ABC be an isosceles triangle in which AB and AC are equal. Produce BC beyond C to any point G. Let F be any point in CG (or prolongation of BC).



From F draw FK perpendicular to AB, and FD perpendicular to AC produced. From C draw CL and CH perpendiculars to AB and FK respectively.

Proof —Because KH and LC are perpendiculars to AB

$\therefore$ HK and CL are parallel (proved in Ex. 2 on page 41)

Similarly, CH and KL or AB are parallel

$\therefore$ the figure KLCH is a parallelogram, so that KH= LC (Theor. 21)

In the $\triangle$ ABC, AB=AC (given)

$\therefore$ the $\angle$ ABC=the $\angle$ ACB (Theor 5)

Because AB and CH are parallel and BG meets them

$\therefore$ the $\angle$ ABC=the $\angle$ HCG (Theor. 13).

$\therefore$ the $\angle$ HCF=the $\angle$ ACB=the vertically opp. $\angle$ DCF

Now, in the two $\triangle^s$ HCF and CDF

Because $\begin{cases} \text{the } \angle \text{CHF= the } \angle \text{CDF (being rt. } \angle^s) \\ \text{the } \angle \text{HCF= the } \angle \text{DCF (proved)} \\ \text{and CF is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, HF=DF

Also, KH=CL (proved)

$\therefore$ CL=KF$-$FH

=KF$-$FD

KF may be greater or less than FD according as F is on BC produced beyond C, or on CB produced beyond B

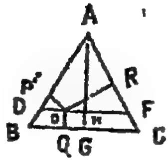
$\therefore$ CL=FK$\sim$FD

Similarly, by drawing perpendicular from B on AC it can be proved that the perpendicular is equal to the difference of FK and FD.

The modification in the foregoing exercise is that the perpendicular drawn either from B on AC or from C on AB

is equal to the difference of the perpendiculars drawn from any point in the prolongation of BC on equal sides AB, AC.

15. Let ABC be an equilateral triangle and AG the perpendicular drawn from A on BC. Let O be any point within the triangle ABC, and let OP, OQ, OR be perpendiculars to AB, BC, CA respectively.



It is required to prove that AG is equal to the sum of OP, OQ and OR.

From O draw DOHF parallel to BC, cutting AB, AG and AC at D, H and F respectively.

Proof.—Because DF and BC are parallel and AB meets them

$\therefore$ the $\angle$ ADF=the $\angle$ ABC. (Theor. 13)

Again, because DF and BC are parallel and AC meets them

$\therefore$ the $\angle$ AFD=the $\angle$ ACB (Theor. 13)

But the $\angle$ ABC=the $\angle$ ACB (being angles of an equilateral triangle)=60°

$\therefore$ the $\angle$ ADF=the $\angle$ AFD=60°

$\therefore$ the $\triangle$ ADF is an equilateral triangle.

Because DH and BG are parallel and AG meets them

$\therefore$ the $\angle$ DHA=the $\angle$ AGB=90°

In the equilateral $\triangle$ ADF the sum of the perpendiculars OP and OR drawn from O in DF on AD, AF respectively, is equal to the perpendiculars drawn either from D on AF or from F on AD (proved in Ex. 14)

But in an equilateral triangle the perpendiculars drawn from the angular points to the opposite sides are equal

$\therefore$ the perpendicular AH is equal to the sum of the perpendiculars OP and OR.

Because OQ and HG are perpendiculars

∴ OQ and HG are parallel (proved in Ex. 2 on page 41).

In the quadrilateral OHGQ, OQ and HG are parallel (proved), and OH and QG are parallel (by construction)

∴ the figure OHGQ is a parallelogram

so that, OQ=HG (Theor. 21)

Because AH=OP+OR, and HG=OQ

∴ AH+HG=OP+OR+OQ

or, AG=OP+OQ+OR

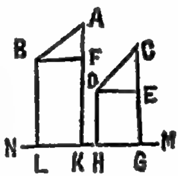
Similarly, it can be proved that each of the perpendiculars drawn from B, C on AC, AB is equal to the sum of OP, OQ and OR.

But AG, being a perpendicular from an angular point of an equilateral triangle to the opposite side, is constant.

∴ (OP+OQ+OR) is also constant wherever the point O may be taken within the triangle ABC.

Q. E. D.

16. Let AB and CD be two equal and parallel straight lines, and NM a third straight line on which perpendiculars BL, AK, DH, CG are drawn from B, A, D, C respectively. Then LK and HG are the projections of BA, DC on the straight line NM.



It is required to prove that LK and HG are equal.

From B draw BF parallel to LK meeting AK in F and from D draw DE parallel to HG meeting CG in E

Proof—Because AK, BL, DH, CG are perpendiculars to the same straight line NM

∴ AK, BL, DH, CG are parallel to one another (proved Ex. 2 on page 41)

Again, because BA is parallel to DC, and AF is parallel to CE

∴ the ∠ BAF = the ∠ DCE

Because LK and BF are parallel and AK meets them

∴ the ∠ BFA = the ∠ LKA = 90°

Because DE and HG are parallel and CG meets them

∴ the ∠ DEC = the ∠ HGC = 90°

Now, in the two $\triangle^s$ ABF and CDE

Because $\begin{cases} \text{the } \angle \text{ BAF = the } \angle \text{ DCE (proved)} \\ \text{the } \angle \text{ AFB = the } \angle \text{ CED (being rt. } \angle^s) \\ \text{and AB = CD (given)} \end{cases}$

∴ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, BF = DE.

In the quadrilateral BFKL, BF is parallel to LK (by contruction) and BL parallel to FK (proved)

∴ BFKL is a parallelogram

so that, BF = LK (Theor. 21)

Similarly, the quadrilateral DHGE is a parallelogram so that, DE = HG (Theor. 21)

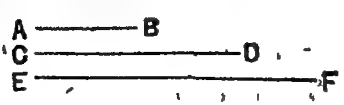
But BF = DE (proved)

∴ LK = HG.

Q. E. D.

Page 68.

1. By means of a diagonal scale draw the straight lines AB = 1·25″, CD = 2 72″ and EF = 3 08″.



2. By means of a diagonal scale draw a straight line AB = 2·68″ long. Measure it in centimetres and millimetres; it will be found to be 6·80 cm. long.

3 By means of a diagonal scale draw a straight line AB=5·7 cm. in length. Measure it in inches and it will be found to be 2·24″ long

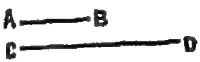
By calculation AB=5·7 × ·3937″=2·24″ nearly.

4. Draw by means of a diagonal scale a straight line AB=3·15″ long. Measure it in centimetres and millimetres and it will be found to be 8 cm. long.

8 cm =3·15″

$\therefore$ 1 cm $=\dfrac{3{\cdot}15}{8}=$ 39″ (correct to two decimal places)

5. Draw by means of a diagonal scale straight lines AB=2·9 cm and CD =6·2 cm. Measure AB and CD in inches and it will be found that AB=1·14″ and CD=2·44″



1·14″=2·9 cm. ; $\therefore$ 1″$=\dfrac{2{\cdot}9}{1{\cdot}14}=$2·54 cm.

2·4 ″=6·2 cm. ; $\therefore$ 1″$=\dfrac{6{\cdot}2}{2{\cdot}44}=$2·54 cm.

$\therefore$ average=$\frac{1}{2}$ (2·54+2·54) cm =2·54 cm.

6. A distance of 100 miles is represented by 1″

$\therefore$ a distance of 1 mile is rep resented by $\frac{1}{100}$ inch

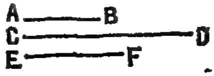
$\therefore$ a distance of 336 miles is represented by $\frac{336}{100}$, or 3·36″

$\therefore$ a distance of 408 miles is represented by $\frac{408}{100}$, or 4·08″

Hence, draw by means of a diagonal scale straight lines AB=3·36″ and CD=4·08″.

Then AB represents 336 miles and CD 408 miles.

7. 1 kilometre or 1000 metres represented by 1 inch



∴ 1 metre is represented by $\frac{1}{1000}$ inch

∴ 850 metres are represented by $\frac{850}{1000}$, or ·85″

∴ 2980 metres are represented by $\frac{2980}{1000}$, or 2·98″

∴ 1010 metres are represented by $\frac{1010}{1000}$, or 1·01″

Hence, draw by means of a diagonal scale straight lines AB= 85″, CD=2 98″ and EF=1·01″.

Then AB represents 850 metres, CD 2980 metres, and EF 1010 metres.

8. 100 links are represented by 1″

∴ 1 link is represented by $\frac{1}{100}$ inch

∴ 417 links are representod by $\frac{417}{100}$, or 4 17″

Hence, by means of a diagonal scale draw a straight line AB=4 17″ long.

Then AB represents 417 links. Measure it in centimetres and millimetres, and it will be found to be 10·6 cm.

9. 5 kilometres are represented by 1 cm.

∴ 42·500 kilometres are represented by $\frac{42\cdot500}{5}$, or 8·500 cm.

Hence, draw by means of a diagonal scale a straight line AB=8 500 cm.

Then AB represents 42·500 kilometres.

Measure it in inches to the nearest hundreths, and it will be found to be 3·35″ long.

10 Because a straight line equal to $2 \cdot 75''$ represents 55 miles

$\therefore$ $1''$ represents $\frac{55}{2 \cdot 75}$, or 20 miles.

Again, because 1 cm. $= \cdot 3937''$

$\therefore$ $1'' = \frac{1}{\cdot 3937}$ cm.

also 1 kilometre $= \frac{5}{8}$ miles nearly

$\therefore$ 1 mile $= \frac{8}{5}$ kilometres.

$\therefore$ $\frac{1}{\cdot 3937}$ cm. represents $20 \times \frac{8}{5}$, or 32 kilometres

$\therefore$ 1 cm. represents $32 \times \cdot 3937$, or $12 \cdot 6$ kilometres nearly.

11. Since $1''$ represents 35 miles.

$\therefore$ $4 \cdot 2''$ represent $35 \times 4 \cdot 2$, or 147 miles (accurately)

$\therefore$ True distance $=147$ miles.

Because 1 kilometre $= \frac{5}{8}$ miles nearly

$\therefore$ 1 mile $= \frac{8}{5}$ kilometres

$\therefore$ $4 \cdot 2''$ represent $147 \times \frac{8}{5}$, or 235 kilometres (approximately)

$\therefore$ approximate distance $=235$ kilometres.

also, $1''$ represents $35 \times \frac{8}{5}$, or 56 kilometres

But $1'' = \frac{1}{\cdot 3937}$ cm.

$\therefore$ $\frac{1}{\cdot 3937}$ cm. represents 56 kilometres

$\therefore$ 1 cm. represents 22 kilometres nearly.

$\therefore$ The scale used in metric measure is 1 cm. representing 22 kilometres nearly.

12. $2\frac{1}{2}''$ or $\frac{5}{2}''$ represent $37\frac{1}{2}$ or $\frac{75}{2}$ miles.

$\therefore$ 1″ represents $\frac{75\times2}{2\times5}$, or 15 miles.

7 cm. represent 88 kilometres

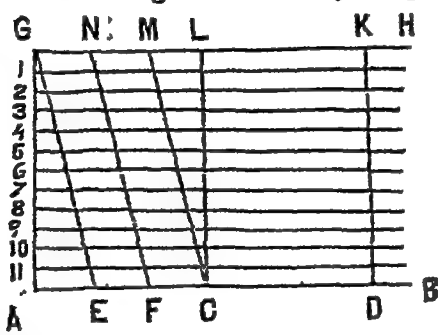
$\therefore$ 1 cm. represents $\frac{88}{7}$ kilometres.

But 1 cm. $=\cdot3937''$ and 1 kilometre $=\frac{5}{8}$ mile nearly.

$\therefore$ $\cdot3937''$ represents $\frac{88}{7}\times\frac{5}{8}$, or $\frac{55}{7}$ miles.

$\therefore$ 1″ represents $\frac{55}{7\times\cdot3937}$, or 20 miles (nearly)

13. Let AB be a straight line of any length. From AB cut off AC, CD,......each equal to 2 cm.



Then AC, CD,......each represents one yard.

From A draw AG (of any length) perpendicular to AB.

From G draw GH parallel to AB. From C, D,..... draw CL, DK,..... perpendiculars to AB meeting GH in L, K,......

Divide AC into three equal parts by E and F. Divide GL into three equal parts by N, M. Join MC, NF, GE.

Then AE, EF, FC each represents one foot.

Divide AG into twelve equal parts by 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. Through these points draw parallels to AB. These parts represent inch divisions.

The scale is thus complete.

In the $\triangle$ MCL, ML represents 1 foot or 12 inches, 1st parallel 11 inches, 2nd parallel 10 inches, and so on, C represents 0 inch.

In the trapezium MCDK, CD represents 1 yard, 11th parallel 1 yd 1 in., 10th parallel 1 yd. 2 in , and so on, MK represents 1 yd. 1 ft.

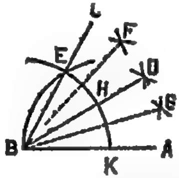
In the trapezium NFDK, FD represents 1 yd. 1 ft., 11th parallel 1 yd 1 ft 1 in., 10th parallel 1 yd. 1 ft. 2 in., and so on, NK represents 1 yd. 2 ft.

In the trapezium GEDK, ED represents 1 yd. 2 ft., 11th parallel 1 yd. 2 ft. 1 in., 10th parallel 1 yd. 2 ft. 2 in., and so on, GK represents 2 yds

In the △ GAE, AE represents, 1 ft., 11th parallel 11 in. 10th parallel 10 in., 9th parallel 9 in., and so on.

Page 79.

1. It is required to construct an angle of 60°, and to divide it into four equal parts.



Construction.—Take a straight line AB of any length.

Take any point K in AB With centres B and K, and radii equal to BK draw two arcs cutting at E. Join BE and produce it to any point C

Then CBA is the required angle of 60°.

With the centres E, K, and radii of equal lengths draw two arcs cutting at D

Join BD, cutting the arc EK at H.

Then the ∠CBA is bisected by BD (Prob. 1)

With the centres H, E, and radii of equal lengths draw two arcs cutting at F. Join BF.

Then the ∠CBD is bisected by BF (Prob 1)

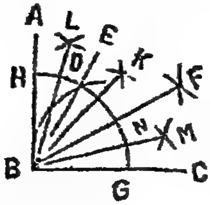
With the centres H, K, and radii of equal lengths draw two arcs cutting at G. Join BG.

Then the ∠DBA is bisected by BG (Prob. 1)

∴ the ∠CBA is divided into four equal parts, by BF, BD and BG.

Q. E. F.

2. It is required to trisect a right angle.



Let ABC be a right angle

With the centre B and radius of any length draw an arc cutting AB, BC at H, G respectively.

With the centre G and radius GB draw another arc cutting the former arc HG at D. Join BD and produce it to any point E.

Then the $\angle$ EBC is 60°

$\therefore$ the $\angle$ ABE $= 90^\circ - 60^\circ = 30^\circ$.

With the centres D and G, and radii of equal lengths draw two arcs cutting at F. Join BF

Then the $\angle$ EBC is bisected by BF (Prob 1)

Because the $\angle$ EBC$=60^\circ$

$\therefore$ each of the $\angle^s$ EBF, FBC$=30^\circ$

also the $\angle$ ABE$=30^\circ$

$\therefore$ the rt $\angle$ ABC is trisected by BE, BF.

Suppose BF cuts the arc HG at N

With the centres H, D, and radii of equal lengths draw two arcs cutting at L. Join BL

Then the $\angle$ ABE is bisected by BL (Prob 1)

With the centres N, D, and radii of equal lengths draw two arcs cutting at K Join BK.

Then the $\angle$ EBF is bisected by BK (Prob 1)

With the centres, N, G, and radii of equal lengths draw two arcs cutting at M Join BM.

Then the $\angle$ FBC is bisected by BM (Prob. 1)

But each of the $\angle^s$ ABE, EBF and FBC$=30^\circ$,

and the $\angle^s$ ABE, EBF and FBC are bisected by BL, BK and BM respectively,

$\therefore$ each of the $\angle^s$ ABL, LBE, EBK, KBF, FBM and MBC$=15°$.

$\therefore$ the $\angle$ KBC$=45°$

$\therefore$ the $\angle$ KBC is trisected by BF, BM.

It is required to show how to trisect an angle of $45°$.

Let KBC be an angle of $45°$.

At B make the $\angle$ CBE$=60°$, $\therefore$ the $\angle$ EBK$=60°-45°=15°$

Bisect the $\angle$ EBC by BF (by Prob. 1)

$\therefore$ each of the $\angle^s$ EBF, FBC$=30°$

The $\angle$ EBF$=30°$ and the $\angle$ EBK$=15°$

$\therefore$ the $\angle$ KBF$=30°-15°=15°$

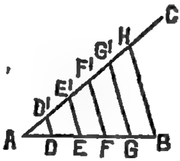
Bisect the $\angle$ FBC by BM (by Prob 1)

$\therefore$ each of the $\angle^s$ FBM, MBC$=15°$

$\therefore$ the $\angle$ KBC of $45°$ is trisected by BF, BM

Q. E. F.

3. Draw a line AB$=6.7$ cm. long.



It is required to divide it into five equal parts.

At A make any convenient angle BAC

From AC mark off five equal parts of any length AD′, D′E′, E′F′, F′G′, G′H

Join HB and from G′, F′, E′, D′ draw G′G, F′F, E′E, D′D parallels to HB meeting AB in G, F, E, D respectively.

Then AB is divided into five equal parts at the points D, E, F and G (Prob. 7).

Measure one of the parts, say AD, in inches and it will be found to be 53″

AB$=6.7$ cm

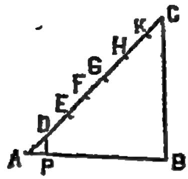
$\therefore$ each of the five equal divisions$=\dfrac{6.7}{5}=1.34$ cm.

But 1 cm. $=\cdot3937$ inch

$\therefore$ 1·34 cm. $=1\cdot34\times\cdot3937''=\cdot53''$ nearly.

Q. E. F.

4. Draw a line $AB=3.72''$ long.



It is required to cut off one-seventh part of it.

At A make any convenient angle BAC.

From AC cut off seven equal parts AD, DE, EF, FG, GH, HK and KC. Join BC. From D draw DP parallel to BC meeting AB in P

Then AP is one-seventh part of AB (Prob. 7)

Measure AP in centimetres and nearest millimetres, and it will be found to be 1·3 cm.

$AB=3\cdot72''$.

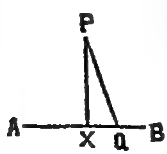
$\therefore AP=\frac{1}{7}.\ AB=\frac{1}{7}\times3\cdot72''$

But 1 inch $=\frac{1}{3937}$ cm.

$\therefore\ AP=\frac{1}{7}\times3\cdot72\times\frac{1}{3937}=1\cdot3$ cm. nearly.

Q. E. F.

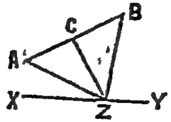
5. Let AB be a straight line, and X any point in it at which a perpendicular $XP=1\ 8''$ is drawn.



With the centre P, and radius $=3''$ draw an arc cutting AB at Q. Join PQ Then PQ is the required oblique.

Measure XQ and it will be found to be $2\frac{1}{4}''$ long.

6. Let XY be the given straight line, and let A and B be the two given points.



It is required to find a point in XY which is equidistant from A and B.

Construction —Join AB and bisect it in C (by Prob 2)

At C draw CZ perpendicular to AB meeting XY in Z. Then Z is the required point.

Join AZ and BZ

Proof—In the two $\triangle^s$ ACZ and BCZ

Because $\begin{cases} AC=BC \text{ (by construction)} \\ CZ \text{ is common to both} \\ \text{and the } \angle ACZ = \text{the } \angle BCZ \text{ (being rt } \angle^s) \end{cases}$

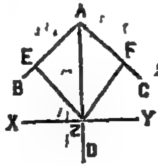
$\therefore$ two $\triangle^s$ are equal in all respects, (Theor. 4)

so that, AZ=BZ

Q E F

This problem is impossible when the two given points A and B are so situated that the straight line CZ bisecting AB at right angles is parallel to XY.

7. Let XY be a straight line, and let AB,AC be two lines intersecting at A.



It is required to find a point in XY which is equistant from AB and AC

Construction.—Bisect the $\angle$BAC (by Prob. 1) by the straight line AD cutting XY in Z.

Then Z is the required point.

From Z draw ZE,ZF perpendiculars to AB,AC respectively.

Proof.—In the two $\triangle^s$ AEZ and AFZ

because { AZ is common to both
the $\angle$ EAZ=the $\angle$ FAZ (by construction)
and the $\angle$ AEZ=the $\angle$ AFZ (being rt $\angle^s$)

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

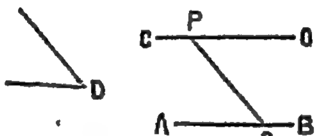
so that, EZ=FZ

$\therefore$ Z is equidistant from two intersecting lines AB and AC

Q. E. F.

This problem is impossible when the two intersecting lines AB, AC are so situated that the line AD bisecting the $\angle$ BAC is parallel to XY.

8. Let AB be the given straight line, P the given point and and D the given angle.



It is required to draw from the point P a straight line PQ making with AB an angle equal to the $\angle$ D

Construction —Through P draw CPO parallel to AB. At P make an angle OPQ=the $\angle$ D, the arm PQ meeting AB in Q

Then PQ is the required line.

Proof.—Because CO and AB are parallel, and PQ meets them

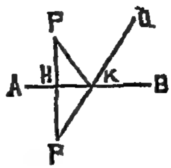
$\therefore$ the $\angle$ OPQ=the alternate $\angle$ PQA (Theor. 14)

But the $\angle$ OPQ=the $\angle$ D (by construction)

$\therefore$ the $\angle$ PQA=the $\angle$ D.

Q. E. F.

9. Let AB be a straight line, and P, Q two given points on the same side of AB.



It is required to draw two lines from P and Q which meet in AB and make equal angles with it.

Construction —From P draw PH perpendicular to AB and produce PH to P′, making HP′ = PH Join P′Q cutting AB at K Join PK.

Then PK and QK are the required lines.

Proof.—In the two $\triangle^s$ PHK and P′HK

because $\left\{\begin{array}{l}\text{PH} = \text{P}'\text{H (by construction)} \\ \text{HK is common to doth} \\ \text{and the } \angle \text{ PHK} = \text{the } \angle \text{ P}'\text{HK (being rt. } \angle^s)\end{array}\right.$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, the $\angle$ PKH = the $\angle$ P′KH

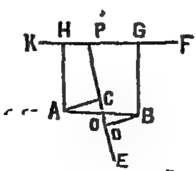
But the $\angle$ P′KH = the vertically opp. $\angle$ QKB (Theor. 3)

$\therefore$ the $\angle$ PKH or the $\angle$ PKA = the $\angle$ QKB.

i. e the lines PK, QK meeting AB in K make equal $\angle^s$ PKA and QKB with AB

Q. E. F.

10 Let P, A, B be the three given points.



It is required to draw a straight line through the given point P such that the perpendiculars drawn to it from A and B are equal.

(i) Construction.—Join AB. Through P draw KPF parallel to AB. From A and B draw AH, BG perpendiculars to KF.

Then KPF is the required line.

Proof—The $\angle$ AHG=1 rt. $\angle$, also the $\angle$ BGH=1 rt. $\angle$ (by construction)

$\therefore$ $\angle^s$ AHG + BGH=2 rt. $\angle^s$

$\therefore$ AH and BG are parallel (Theor. 13)

In the quadrilateral ABGH, AH is parallel to BG (proved) and HG parallel to AB (by construction)

$\therefore$ ABGH is a parallelogram.

$\therefore$ AH=GB (Theor. 21)

i. e. the perpendiculars AH and GB drawn from A, B respectively to the straight line KF passing through P are equal.

$\therefore$ KF is the required line.

(ii) Construction—Bisect AB at O. Join PO and produce it to any point E. From A and B draw AC and BD perpendiculars to PE.

Then PE is the required line.

Proof—In the $\triangle^s$ AOC and BOD

Because $\begin{cases} \text{AO=BO (by construction)} \\ \text{the } \angle \text{ AOC=the } \angle \text{ BOD (Theor 3)} \\ \text{and the } \angle \text{ ACO=the } \angle \text{ BDO (being rt. } \angle^s) \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, AC=BD

i e the perpendiculars AC, BD drawn from A, B respectively to the straight line PE are equal.

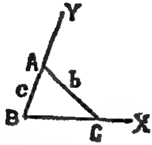
$\therefore$ PE is the required line.

Q. E F.

This problem is not always possible It is impossible when the three points A, B P are in the same straight line, and P does not lie at the middle point of the line joining A and B.

Page 82

(i) To construct a triangle ABC when the angle B is given and b is greater than c.

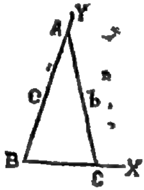


Construction—Make an angle YBX = the given $\angle$B. From BY cut off a part BA $= c$ With the centre A and radius $= b$, draw an arc cutting BX at C. Join AC.

Then ABC is the required triangle.

Only one triangle can be constructed with the given parts The nature of the triangle ABC depends on the lengths of the given sides and the magnitude of the given angle

(ii) To construct a triangle ABC when the angle B is given and b is equal to c.

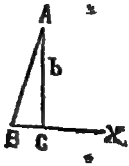


Construction—Make an angle YBX = the given $\angle$ B. From BY cut off a part BA $= c$ With the centre A and radius $= b$ draw an arc cutting BX at C. Join AC.

Then ABC is the required triangle.

Only one triangle can be constructed with the given parts which has $\angle$ B $= \angle$ C, so that the figure is an isosceles triangle.

(iii) To construct a triangle ABC when angle B is given, c is given and b is equal to the perpendicular from A to the opposite side BC.

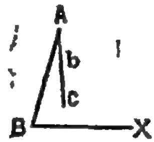


Construction.—Make an angle ABX= the given ∠B making the arm BA=c. With the centre A and radius $=b$ draw an arc touching BX at C. Join AC.

Then ABC is the required triangle.

Only one triangle right-angled at C can be constructed with the given parts.

(iv) To construct a triangle ABC when angle B is given, c is given and b is less than the perpendicular from A on BC.

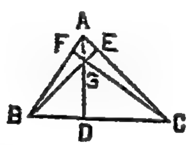


Construction —Make an angle ABX= the given ∠B making the arm BA=c. With the centre A and radius $=b$ draw an arc This arc does not touch the side BC

Hence no triangle can be constructed with the given parts.

Page 84.

1. It is required to construct a triangle whose sides are 7·5 cm., 6 2 cm , and 5·3 cm.



Make a straight line BC=7·5 cm. With the centres B and C, and radii equal to 5 3 and 6·2 cm respectively draw two arcs cutting at A. Join BA and CA

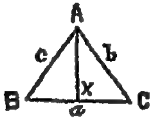
Then ABC is the required triangle.

Draw AD perpendicular from A to BC, CF perpendicular from C to AB, and BE perpendicular from B to AC. These perpendiculars cut at G.

Measure AD, CF and BE, and it will found that AD= 5 2 cm., CF=6·1 cm. and BE=4·3 cm.

Q. E. F.

2. It is required to draw a triangle ABC, having given $a=3''$, $b=2·5''$ and $c=2·75''$



Take a straight line BC$=a$ With the centres B and C and radii equal to c and b respectively draw two arcs cutting at A Join BA and CA.

Then ABC is the required triangle.

Bisect the $\angle$ BAC (Prob. 1) by the straight line AX meeting BC in X

Measure BX and XC, and it will be found that BX= 1 57″ and CX=1·43″

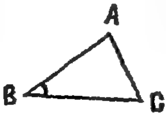
$$\frac{BX}{CX}=\frac{1\ 57}{1·43}=1\ 0979=1\ 10 \text{ (correct to two decimal places)}$$

and $\dfrac{c}{b}=\dfrac{2·75}{2\ 5}=1·10$. These two results allowing for probable errors in measurement may be regarded as identical

$$\therefore \frac{BX}{CX}=\frac{c}{b}.$$

Q. E F.

3 It is required to draw a plan (1 inch to 100 yds.) of a triangular field in which two sides are 315 yds., 260 yds., and the included angle=39°.



Construct an angle ABC=39° having the arms AB= 3 15″ and BC=2 6″. Join AC

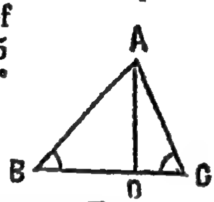
Then ABC is the required triangular field.

Measure AC and it will be found to be 2″ long.

Hence the side AC or the remaining side of the field = 200 yards.

Q E F.

4. It is required to draw a plan (1 cm. to 10 metres) of a triangular plot of ground ABC of which the base BC is 75 metres and the angles at B and C are 47° and 68° respectively.



Take BC = 7 5 cm. At B make an angle = 47° and at C make an angle = 68° the two arms meeting at A. Then ABC is the required triangular plot.

The $\angle^s A+B+C$ of the $\triangle ABC=180°$ (Inf. 1 Theor. 16)

But the $\angle B=47°$ and $\angle C=68°$

$\therefore$ the $\angle A=180°-(68°+47°)$

$=180°-115°=65°$.

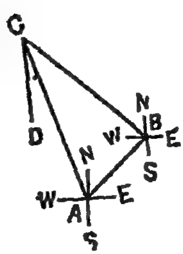
Measure AB and AC and it will be found that AB = 7·7 cm. and AC = 6 1 cm.

Hence the other sides of the triangular field are 77 metres and 61 metres long.

From A draw AD perpendicular to BC and measure it. It will be found that AD = 5 6 cm. Hence the perpendicular from A to BC is 56 metres.

Q. E. F.

5. It is required to draw a chart of the whole course of a yacht (2 cm. to 1 knot) which steers first N. E. for 20 min, and then N. W for 35 min., sailing at an average speed of 9 knots per hour.



In 1 hr. the yacht steers 9 knots.

or, in 60 min the yacht steers 9 knots

$\therefore$ in 1 min. the yacht steers $\frac{9}{60}$ knots

$\therefore$ in 20 min. the yacht steers $\frac{9}{60}\times 20$, or 3 knots

and in 35 min. the yacht steers $\frac{9}{60}\times 35$, or $\frac{21}{4}$ knots.

From any point A draw AB in N. E. direction making AB=6 cm From B draw BC in N W direction making BC=$\frac{21}{4}$ cm

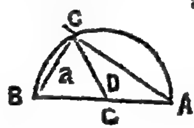
Then this graph represents the course of the yacht. Join AC and measure it. It will be found to be 12 08 cm.

Hence, the yacht is 6 04 knots from the harbour.

From C draw CD vertically downwards. Measure the $\angle$ ACD and it will be found to be 15°. Hence the yacht must set a course 15° E. of south for the run home.

Q. E F.

6. It required to draw a triangle having given that the hypotenuse c=10 6 cm and side a=5·6 cm.



Take a straight line BA=c Bisect it at D (Prob 2).

With the centre D and radius DB or DA describe a semi-circle. With the centre B and radius=a draw an arc cutting the semi-circle at C Join AC and BC

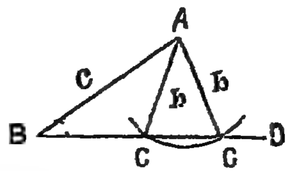
Then ABC is the required right-angled triangle, right-angled at C (Prob. 10)

Measure AC, and it will be found to be 9 cm. long.

$\sqrt{c^2-a^2}=\sqrt{(10\ 6)^2-(5\ 6)^2}=\sqrt{81}=9$

$\therefore b=\sqrt{c^2-a^2}$

7. It is required to construct a triangle having given the following parts :— $B=34°$, $b=5·5$ cm., $c=8·5$ cm.



Take a straight line BD of any length. At B make the $\angle$ DBA$=34°$ making the arm BA$=c$.

With the centre A and radius equal to b draw an arc : this arc cuts BD at two points C′ and C.

Join AC′ and AC.

Hence there are two triangles ABC and ABC′ satisfying the given condition.

Measure BC′ and BC, and it will be found that BC′$=$ 4·3 cm. and BC$=$9·8 cm.

Thus the two values of a are 4·3 cm. and 9·8 cm.

Measure also the $\angle^s$ ACB and AC′B and it will be found that the $\angle$ ACB$=60°$ and the $\angle$ AC′B$=120°$. Thus the two values of C are $60°$ and $120°$. Because $60°+120°=180°$, hence the two values of C are supplementary.

Because AC$=$AC′

$\therefore$ the $\angle$ ACC′$=$the $\angle$ AC′C (Theor. 5)

But the $\angle^s$ AC′B and AC′C together$=2$ rt. $\angle^s$ (Theor. 1)

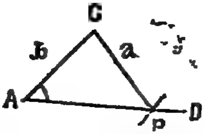
$\therefore$ the $\angle^s$ AC′B and ACC′ or ACB together$=2$ rt. $\angle^s$ *i. e.* the $\angle^s$ AC′B and ACB are supplementary.

Q. E. F.

8. It is required to construct a triangle ABC having given the angle $A=50°$, $b=6·5$ cm. and a.

Take a straight line AD of any length. At A make the $\angle$ DAC$=50°$ making AC$=b$.

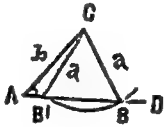
(*i*). When $a=7$ cm.



With the centre C and radius$=7$ cm. draw an arc cutting AD at B. This arc cuts AD at only one point. Join CB

Then ABC is the required triangle.

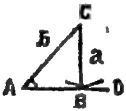
(*ii*). When $a=6$ cm.



With the centre C and radius$=6$ cm. draw an arc. This arc cuts AD at two points B and B′. Join CB and CB′.

Then ABC and AB′C are the required triangles, satisfying the given condition.

(*iii*). When $a=5$ cm.

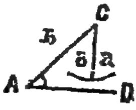


With the centre C and radius$=5$ cm draw an arc. This arc touches AD at a point B

Join BC

Then ABC is the required triangle right-angled at B.

(*iv*). When $a=4$ cm.

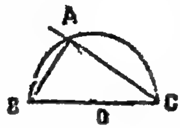


With the centre C and radius $=4$ cm. draw an arc,

This arc does not cut nor touch the side AD,

Hence, no triangle exists with the given parts.

9 Let AB, AC be two straight rods crossing at right angles at A, and carried over a straight canal by bridges at B and C, such that the distance between the bridges is 461 yds, and the distance from the crossing A to the bridge B is 261 yds.



It is required to draw a plan and ascertain the distance from A to C by measurement.

Draw the plan in the scale of 1 cm. to 100 yds.

Take a straight line BC$=4\cdot61$ cm. and upon BC as diameter describe a semi-circle With the centre B and radius$=2\cdot61$ cm draw an arc cutting the semi-circle at A. Join CA and BA

Measure AC and it will be found $3\cdot8$ cm. long.

$\therefore$ The distance of C from A is 380 yds.

Q. E. F.

10. (see figure in Ex. 1 on page 19)

It is required to draw an isosceles triangle on a base of 4 cm. and having an altitude of $6\cdot2$ cm.

Construction—Take a straight line BC$=4$ cm.

Bisect it at D (Prob. 2). At D draw DA perpendicular to BC making DA$=6\cdot2$ cm. Join BA and CA.

Then ABC is the required isosceles triangle.

Proof.—In the two $\triangle^s$ ABD and ACD

Because $\begin{cases} \text{BD}=\text{DC (by construction)} \\ \text{DA is common to both} \\ \text{and the } \angle \text{ ADB}=\text{the } \angle \text{ ADC (being rt. } \angle^s) \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

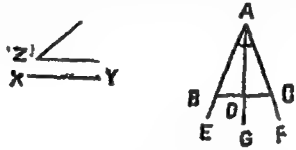
so that, AB$=$ AC.

∴ ABC is an isosceles triangle, having the altitude AD $=6\cdot2$ cm. and base BC $=4$ cm.

Measure AB and AC, and it will be found that each of them $=6\cdot5$ cm

Q. E. F.

11 Let Z be the given angle and XY the given straight line.



It is required to draw an isosceles triangle having its vertical angle equal to the given angle Z and the perpendicular from the vertex on the base equal to the given straight line XY.

Construction.—Construct an angle EAF $=$ the $\angle$ Z Bisect the $\angle$ EAF (Prob. 1) by AG. From AG cut off AD $=$ XY. At D draw DB perpendicular to AG meeting AE in B Produce BD to meet AF in C.

Then ABC is the required isosceles triangle.

Proof —In the two $\triangle^s$ ABD and ACD

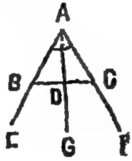
because $\begin{cases} \text{the } \angle \text{ BAD} = \text{the } \angle \text{ CAD (by construction)} \\ \text{the } \angle \text{ ADB} = \text{the } \angle \text{ ADC (being rt } \angle^s) \\ \text{and AD is common to both} \end{cases}$

∴ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, AB $=$ AC

∴ ABC is an isosceles triangle having the altitude AD $=$ XY, and the vertical angle BAC $= \angle$ Z

(2) It is required to construct an equilateral triangle in which the perpendicular from one vertex on the opposite side is 6 cm.



We know that in an equilateral triangle all its sides are equal. Hence all its angles are equal. Therefore each of its angles $= \frac{1}{3}$ of $180^\circ = 60^\circ$.

Construction.—Make an angle $EAF = 60^\circ$.

Bisect the $\angle$ EAF by AG (Prob. 1). From AG cut off $AD = 6$ cm. At D draw DB perpendicular to AG meeting AE in B Produce BD to meet AF in C.

Proof.—In the two $\triangle^s$ ABD and ACD

Because $\begin{cases} \text{the } \angle \text{ BAD} = \text{the } \angle \text{ CAD (by construction)} \\ \text{the } \angle \text{ ADB} = \text{the } \angle \text{ ADC (being rt. } \angle^s) \\ \text{and AD is common to both} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 17)

so that, the $\angle$ ABD = the $\angle$ ACD

Now, the $\angle$ BAC $= 60^\circ$

$\therefore \angle ABC + \angle ACB = 180^\circ - 60^\circ = 120^\circ$

But the $\angle$ ABC = the $\angle$ ACB (proved)

$\therefore$ each of the $\angle^s$ ABC and ACB $= \frac{1}{2}$ of $120^\circ = 60^\circ$

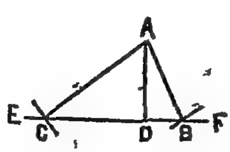
$\therefore$ the $\angle$ ABC = the $\angle$ ACB = the $\angle$ BAC

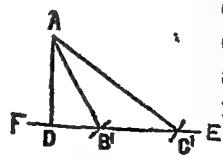
Hence the triangle ABC is equilateral (Theor 6. Cor).

Measure a side and it will be found to be 6·9 cm. long.

Q. E. F.

12. It is required to construct a triangle ABC in which the perpendicular from A on BC is 5 cm., and the sides AB, AC are 5·8 cm. and 9 cm. respectively.





Construction—Take a straight line EF of any length. At D any point in EF draw DA perpendicular to EF making DA = 5 cm. With the centre A and radii = 9 cm., and

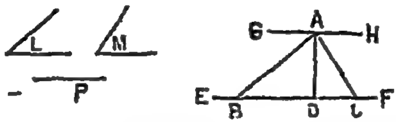
5·8 cm. draw two arcs cutting EF in the points B and C on opposite sides of AD, or in the points B′ and C′ on the same side of AB

Then ABC and AB′C′ are the two required triangles which satisfy the given conditions

Measure BC and B′C′ and it will be found that BC = 10·4 cm. and B′C′ = 4·5 cm.

Q E. F.

13. Let L, M be the given angles and P the given line.



It is required to construct a triangle ABC having the angles at B and C equal to two given angles L and M, and the perpendicular from A on BC equal to the given line P.

Construction —Draw a straight line EF of any length. At D any point in EF draw DA perpendicular to EF making DA = the line P.

Through A draw GAH parallel to EF. At A make the ∠ GAB = the ∠ L the arm AB meeting EF in B. Also at A make the ∠ HAC = the ∠ M the arm AC meeting EF in C.

Then ABC is the required triangle.

Proof —Because GH and EF are parallel, and AB meets them

∴ the ∠ GAB = the alternate ∠ ABC (Theor 14)

= the ∠ L

Again, because GH and EF are parallel, and AC meets them

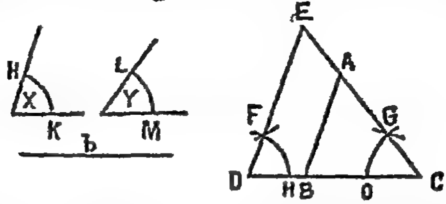
∴ the ∠ HAC = the alternate ∠ ACB (Theor. 14)

= the ∠ M

And the perpendicular AD = the given line P (by construction)

Q. E. F.

14. Let X, Y be the given angles and b the given side.



It is required to construct a triangle ABC having given two angles at B and C respectively equal to the angles X and Y, and the side b.

Construction.—With the vertex X of the $\angle$ X as centre and any radius draw an arc cutting the arms of the angle at H and K. With the vertex Y of the $\angle$ Y as centre and radius of any length draw an arc cutting the arms of the angle at L and M.

Take a straight line DC of any length. With the centre D and radius = XK or XH draw an arc cutting DC at H. With the centre H and radius = HK draw another arc cutting the former arc at F. Join DF. With the centre C and radius = YM or YL draw an arc cutting CD at O. With the centre O and radius = LM draw another arc cutting the former arc at G Join CG

Produce DF and CG to meet in E From CE cut off CA $= b$. Through A draw AB parallel to DE meeting CD in B.

Then ABC is the required triangle.

Proof—Because AB and ED are parallel, and DC meets them

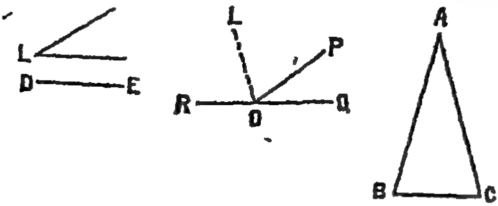
$\therefore$ the $\angle$ EDC = the $\angle$ ABC (Theor. 14)

= the $\angle$ X

also, the $\angle$ ACB = the $\angle$ Y, and side AC $= b$.

Q. E. F.

15 Let L be the given angle and DE the given side.



It is required to construct an isosceles triangle having its vertical angle equal to the given angle L and its base equal to DE.

Construction —Make an angle POQ = the ∠ L. Produce one of the arms QO of the angle to any point R, then the ∠ POR is supplement of the ∠ POQ. Bisect the ∠ POR by OL, then each of the ∠s POL, LOR is half the supplement of the ∠ POQ.

Take a straight line BC = DE. At the points B and C make the ∠s CBA, BCA each equal to the ∠ LOR or ∠ POL the arms BA and CA meeting at A

Then ABC is the required isosceles triangle.

Proof.—The ∠ CBA = the ∠ BCA (by construction)

∴ AB = AC (Theor. 6)

∴ ABC is an isosceles triangle.

Again, the vertical ∠ BAC

= the supplement of the ∠s ABC + ACB

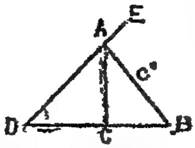
= the supplement of the ∠s POL + LOR

= the ∠ POQ

And the base BC = DE.

Q. E. F.

16. It is required to construct a right angled triangle having given the hypotenuse $c=5{\cdot}3$ cm, and the sum of the remaining two sides a and $b=7\,3$ cm.



Construction.—Take a straight line DB=7·3 cm. At D make the ∠ BDE=half rt. angle or 45°.

With the centre B and radius=5 3 cm. draw an arc cutting DE at A. From A draw AC perpendicular to DB. Join AB

Then ABC is the required right-angle triangle.

Proof.—Because the ∠ ACD is a rt ∠, and the ∠ EDB or ADC=half rt ∠ or 45°.

∴ the ∠ DAC also=half rt. ∠ or 45° (Theor. 16 Inf 3)

∴ the ∠ ADC=the ∠ DAC

∴ DC=AC (Theor 6)

∴ AC+BC=DC+BC=DB=7·3 cm.

also the hypotenuse AB=5·3 cm. (by construction).

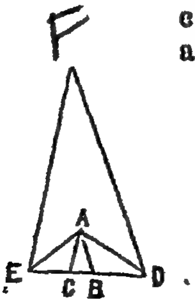
Measure AC and BC, and it will be found AC=b=4·5 cm. and BC=a=2·8 cm.

$\therefore \sqrt{a^2+b^2}=\sqrt{(2{\cdot}8)^2+(4{\cdot}5)^2}=\sqrt{28{\cdot}09}=5{\cdot}3$

$\therefore c=\sqrt{a^2+b^2}$

Q E. F.

17. It is required to construct a triangle having given its perimeter $a+b+c=12$ cm, and the angles at the base equal to 70° and 80° respectively.



Construction.—Take a straight line ED=12 cm. At D make an angle EDF=70°, and at E make an angle DEF =80°, the two arms EF and DF meeting in F

Bisect the $\angle$EDF by DA and the $\angle$DEF by EA, the two bisectors DA and EA meeting in A

From A draw AB parallel to FD and AC parallel to FE, the lines AB and AC meeting ED in B and C.

Then ABC is the required triangle.

Proof.—Because FE and AC are parallel, and AE meets them

∴ the $\angle$AEF=the alternate $\angle$EAC (Theor. 14)

But the $\angle$AEF=the $\angle$AEC (by construction)

∴ the $\angle$EAC=the $\angle$AEC

∴ AC=EC (Theor. 6)

Again, because AB and FD are parallel, and AD meets them

∴ the $\angle$ADF=the alternate $\angle$DAB (Theor. 14)

But the $\angle$ADF=the $\angle$ADB (by construction)

∴ the $\angle$DAB=the $\angle$ADB

∴ AB=BD (Theor 6)

∴ AC+BC+BA=EC+CB+BD=ED=12 cm.

Again, because AC and FE are parallel, and ED meets them

∴ the $\angle$FED=the $\angle$ACD (Theor. 14)

or, the $\angle$ACB=80°

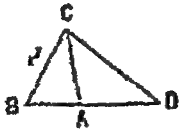
And, because AB and FD are parallel, and DE meets them

∴ the $\angle$FDE=the $\angle$ABE (Theor. 14)

or, the $\angle$ABC=70°.

Q. E. F.

18 It is required to construct a triangle ABC in which $a=6.5$ cm, $b+c=10$ cm, and $B=60°$.



Construction.—Take a straight line DB=10 cm. At B make an angle DBC=60° making BC=6·5 cm Join DC

At C make the ∠DCA=the ∠CDA the arm CA meeting BD in A.

Then ABC is the required triangle.

Proof.—Because the ∠ACD=the ∠ADC (by construction)

∴ AC=AD (Theor. 6)

∴ AC+AB=DA+AB=DB=10 cm.

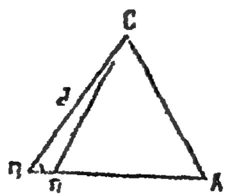
also BC=6·5 cm., and the ∠ABC=60°.

Measure AB or c and AC or b, and it will be found that AB=4·2 cm. and AC=5·8 cm.

∴ $b+c=(5\cdot8+4\cdot2)$ cm $=10$ cm.

Q. E. F.

19. It is required to construct a triangle ABC in which $a=7$ cm., $c-b=1$ cm., and $B=55°$.



Construction.—Take a straight line BD=1 cm. At B make an angle DBC=55° making BC=7 cm.

Join DC. Produce BD to any point A.

At C make the angle $DCA =$ the $\angle ADC$, the arm CA meeting BD produced in A

Then ABC is the required triangle

Proof.—Because the $\angle ADC =$ the $\angle ACD$ (by construction)

$\therefore AD = AC$ (Theor 6)

$\therefore BA - CA = BA - DA = BD = 1$ cm

also $BC = 7$ cm., and the $\angle ABC = 55^\circ$

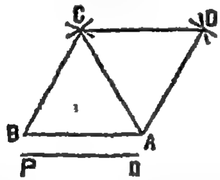
Measure AB or c and AC or b, and it will be found that $AB = 8$ cm and $AC = 7$ cm.

$\therefore c - b = 8$ cm.$- 7$ cm.$= 1$ cm.

Q. E. F.

Page 89.

1. Let PQ be a given straight line.



It is required to draw a rhombus each of whose sides is equal to the given straight line PQ, which is also to be one diagonal of the figure.

Construction.—Take a straight line $BA = PQ$.

With the centres B and A, and radius$= BA$ draw two arcs cutting at C.

Join BC and CA. With the centres C and A, and radius$= BA$ draw two arcs cutting at D on the side of CA opposite to B.

Join CD and AD.

Then ABCD is the required rhombus each of whose sides is equal to PQ, also one diagonal CA is equal to PQ.

In the $\triangle$ ABC, because BA=BC=CA (by construction)

$\therefore$ ABC is an equilateral triangle

$\therefore$ each of its angles BAC, ACB, ABC=60°.

Similarly, in the $\triangle$ ADC, because CA=AD=CD (by construction)

$\therefore$ ADC is an equilateral triangle

$\therefore$ each of its angles ADC, ACD, DAC=60°.

$\therefore$ the $\angle$ BCD=$\angle$ ACB+$\angle$ ACD=60°+60°=120°

Similarly, the $\angle$ BAD=$\angle$ BAC+$\angle$ DAC=60°+60°=120°.

The $\angle$ ABC=60°, also the $\angle$ ADC=60°.

Q. E. F.

2. (See figure in Ex. 3 on page 19)

It is required to draw a square on a side of 2·5 inches and to prove theoretically that its diagonals are equal.

Construction.—Take a straight line DC=2·5″.

At D and C draw DA and CB perpendiculars to DC making each of them equal to 2·5″

Join AB

Then ABCD is the required square.

Join AC and BD.

Then in the two $\triangle^s$ ADC and BCD

Because $\begin{cases} \text{AD=BC (by construction)} \\ \text{DC is common to both} \\ \text{and the } \angle \text{ ADC=the } \angle \text{ BCD (being rt. } \angle^s) \end{cases}$

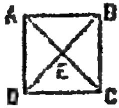
$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, AC=BD

Measure AC and BD, and it will be found that each of them=3·54″.

Q. E. F.

3 It is required to construct a square on a diagonal of 3″.



Construction —Take a straight line AC=3″.

Bisect it at E (Prob 2) At E draw ED perpendicular to AC making ED=½ AC=1·5″. Produce DE beyond E to B making EB=ED.

Join AB BC CD and DA.

Then ABCD is the required square.

Measure the sides, and it will be found that each of them is equal to 2 1.″

The average result=2·12″

Q. E. F.

4. (See figure in Ex. 3 on page 59)

It is required to draw a parallelogram ABCD, having given that one side AB=3 5 cm, and the diagonals AC,BD are 8 cm and 6 cm respectively.

We know that the diagonals of a parallelogram bisect one another (Th or. 21 Cor. 3)

Construction.—Take a straight line AB=3·5 cm.

With the centre A and radius=½ AC or 4 cm. draw an arc. With the centre B and radius=½ BD or 3 cm draw another arc cutting the former arc at O

Join AO and BO Produce AO beyond O to C making OC=AO=4 cm Also produce BO beyond O to D making OD=BO=3 cm.

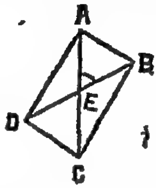
Join AD DC and BC.

∴ AC=8 cm. and BD=6 cm., and they bisect one another at O.

Hence ABCD is the required parallelogram.
Measure AD and it will be found that AD=4·4 cm.

Q. E. F.

5. It is required to construct a quadrilateral whose diagonals are equal (each 6 cm.) and these diagonals bisect one another at an angle of 60°, to name its species and give a *formal* proof.



Construction —Make an angle AEB=60°, making the arms EB,EA each equal to 3 cm.

Produce AE beyond E to C making EC=AE.

Also produce BE beyond E to D making ED=BE.

Join AD, DC, CB and BA

Then ABCD is the required quadrilateral.

This quadrilateral is rectangle.

Proof—In the $\triangle^s$ AEB and DEC

Because $\begin{cases} AE=EC \text{ (by construction)} \\ BE=ED \text{ (by construction)} \\ \text{and the } \angle AEB = \text{the } \angle DEC \text{ (Theor. 3)} \end{cases}$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, AB=DC and the $\angle$ BAE=the $\angle$ DCE

But the $\angle^s$ BAC and DCA are alternate angles

$\therefore$ AB and DC are parallel (Theor. 13)

$\therefore$ AD and BC are equal and parallel. (Theor. 20)

$\therefore$ the figure ABCD is a parallelogram

In the two $\triangle^s$ ABC and DCB

Because $\begin{cases} AB=DC \text{ (proved)} \\ AC=DB \text{ (given)} \\ \text{and BC is common to both} \end{cases}$

$\therefore$ two Δ^s are equal in all respects (Theor. 7)

so that, the $\angle$ ABC = the $\angle$ DCB

But the $\angle^s$ ABC + DCB = 2 rt. $\angle^s$ (Theor. 14)

$\therefore$ each of the $\angle^s$ ABC and DCB = $\frac{1}{2}$ of 2 rt. $\angle^s$ = 1rt. $\angle$

$\therefore$ The parallelogram ABCD is a rectangle.

Q. E. F.

It is required to show that *five* independent data are here given.

Five independent data given in this problem are :—

(1) length of AC, (2) length of BD, (3) AC bisects BD, (4) BD bisects AC, and (5) AC, BD cut at an angle of 60°.

Measure AB, BC, and it will be found that AB = 3 cm., and BC = 5 2 cm.

Hence the perimeter = AB + BC + CD + DA

= (3 + 5 2 + 3 + 5 2) cm.

= 16·4 cm.

It is required to find the increase per cent in the perimeter of the quadrilateral if the angle between the diagonals were increased to 90°.

If the angle between the diagonals were increased to 90°, the figure would be a square, as shown in the figure of Ex. 3.

Measure each side of the square, and it will be found that it is equal to 4·24 cm.

$\therefore$ the perimeter = 16·96 cm.

$\therefore$ increase in perimeter = (16·96 − 16·4) cm. = 56 cm.

$\therefore$ increase per cent. in the perimeter $= \frac{\cdot 56}{16\cdot 4} \times 100 = 3\cdot 4$ nearly

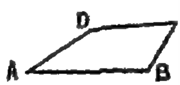
6. In a quadrilateral ABCD it is given that AB=5·6 cm , BC=2 5 cm , CD=4 cm,, and DA=3·3 cm.

It is required to show that the shape of the quadrilateral is not settled by these data.

Because *five* independent data are necessary to construct a quadrilateral, and here are given only four , therefore the shape of the quadrilateral will not be settled unless one more is given.

(i) It is required to draw the quadrilateral when $\angle$ A =30°

Construction—Make an angle DAB=30°, making the arms AB and AD, respectively equal to 5 6 cm. and 3·3 cm. With the centre D and radius =4 cm draw an arc. With the centre B and radius =2 5 cm. draw another arc cutting the former arc at C.

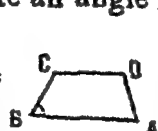


Join DC and BC.

Then ABCD is the required quadrilateral.

(ii) It is required to draw the quadrilateral when $\angle$ A = 60°

Construction—Make an angle BAD=60°, making the arms AB, AD respectively equal to 5 6 cm., and 3 3 cm With the centre B and radius= 2·5 cm draw an arc With the centre D and radius=4 cm. draw another arc cutting the former arc at C.

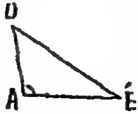


Join BC and DC.

Then ABCD is the required quadrilateral.

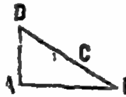
It is required to show why does the construction fail when the $\angle$ A=100°.

If the $\angle$ A were made 100°, then the arcs drawn with centres B, D, and radii equal to 2 5 cm., and 4 cm., will not cut one another, because BD is, in this case, greater than (4+2 5) cm., or 6 5 cm Hence the construction fails, as shown in the figure and no quadrilateral exists.



It is required to determine the least value of the $\angle$A for which the construction fails

The least value of the $\angle$A for which the construction fails is one in which the two arcs, drawn with centres B and D, and radii equal to 2·5 cm, and 4 cm., just touch one another, that is, when BD=(4+2·5) cm, or 6 5 cm



This is the case when the $\angle$ A=90°, as shown in the figure

7 It is required to explain the method of constructing a quadrilateral, having given the lengths of the four sides and that of one diagonal.

In order to construct such a quadrilateral, take a line equal to its diagonal, and on the opposite sides of this line describe two triangles whose sides are equal to the two pair of adjacent sides of the quadrilateral which meet at the angular points through which the diagonal does not pass (according to the method explained in Prob 6)

It is required to find the conditions that must hold among the given data in order that the problem may be possible.

In order that the problem may be possible, each of these pairs of adjacent sides must be together greater than the given diagonal.

(i) It is required to construct a quadrilateral when AB=3″, BC=1·7″, CD=2 5″, DA=2 8″, and the diagonal BD=2,6″.

Construction.—Take a straight line BD$=2{\cdot}6''$

With the centre B and radius$=1{\cdot}7''$ draw an arc. With the centre D and radius$=2{\cdot}5''$ draw another arc cutting the former arc at C.

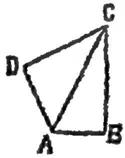
Again with the centre B and radius$=3''$ draw an arc on the side of BD opposite to C. With the centre D and radius$=2{\cdot}8''$ draw another arc cutting the former arc at A.

Join AB, BC, CD and DA.

Then ABCD is the required quadrilateral.

Join AC and measure it. It will be found that AC$=4{\cdot}25''$.

(*iv*) It is required to construct a quadrilateral when AB$=3{\cdot}6$ cm., BC$=7{\cdot}7$ cm., CD$=6{\cdot}8$ cm., DA$=5{\cdot}1$ cm., and the diagonal AC$=8{\cdot}5$ cm.



Construction.—Take a straight line AC$=8{\cdot}5$ cm.

With the centre A and radius$=3{\cdot}6$ cm. draw an arc. With the centre C and radius$=7{\cdot}7$ cm. draw another arc cutting the former arc at B.

Again with the centre A and radius$=5{\cdot}1$ cm. draw an arc on the side of AC opposite to B. With the centre C and radius$=6{\cdot}8$ cm. draw another arc cutting the former arc at D.

Join AB, BC, CD and DA.

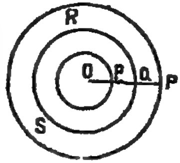
Then ABCD is the required quadrilateral.

Measure the angles B and D, and it will be found that the $\angle$ B$=90^\circ$ and the $\angle$ D$=90^\circ$.

Q. E. F.

Pages 94 and 95.

1. Let QRS be a given circle whose centre is O, and let P be a given point which moves so that its distance (measured radially) from the circumference of the circle QRS is constant.



It is required to find the locus of the point P.

Join OP, and let it cut the given circle at Q.

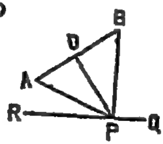
Then since the circle is given, its radius OQ is of constat length Also QP is of constant length (given)

$\therefore$ $(OQ+QP)$ and $(OQ-QP)$ are also constants

Hence the locus of P is a pair of concentric circles whose radii are $(OQ+QP)$ or OP and $(OQ-QP)$ or OP' as shown in the figure

Q E. F.

2. Let a point P move along a straight line RQ, and let A and B be any two given points.



It is required to find the position in which P is equidistant from A and B

Join AB and bisect it at O. At O draw OP perpendicular to AB meeting RQ in P.

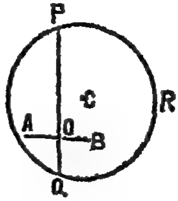
Join AP and BP.

Locus of the points equidistant from A and B is the straight line OP which bisects AB at right angles (Prob. 14)

Hence the point common to OP and RQ must satisfy both the conditions, that is, the point P, where OP intersects RQ, lies on RQ and is equidistant from A and B

Q. E. F.

3. Let PQR be a circle whose centre is C, and let A and B be two fixed points within the circle.



It is required to find points on the circumference of the of the circle PQR equidistant from A and B.

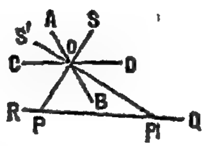
Join AB and bisect it at O. At O draw OP perpendicular to AB meeting the circumference of the circle PQR in P. Produce PO beyond O to Q meeting the circumference of the same circle in Q.

Locus of the points equidistant from A and B is the straight line POQ which bisects AB at right angles (Prob. 14)

Hence the points common to the circle PQR and the straight line PQ must satisfy both the conditions, that is, P and Q, the points of intersection of OP and OQ and the circle PQR, lie on the circumference of the circle and are equidistant from A and B.

Q. E. F.

4. Let a point P move along a straight line RQ, and let AB, CD be two other given straight lines.



It is required to find the position of P which is equidistant from AB and CD.

Let AB, CD cut one another at O.

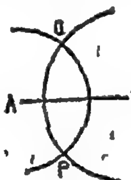
Bisect the ∠ BOC by OP meeting RQ in P. Produce PO beyond O to any point S. Bisect the ∠ BOD by OP' meeting RQ in P'. Produce P'O beyond O to any point S'.

Locus of the points equidistant from AB and CD is the pair of lines SP, S′P′ bisecting the angles between AB and CD (Prob. 15)

Hence the points common to this pair of lines and the given straight line RQ must satisfy both the conditions; that is P, P′, the points of intersection of the pair of lines SP, S′P′, and the straight line RQ, lie on the straight line RQ and are equidistant from AB and CD

Q. E. F

5. Let A and B be two fixed points 6 cm. apart.



It is required to find two points which are 4 cm distant from A, and 5 cm from B

With the centre A and radius=4 cm describe a circle. With the centre B and radius=5 cm. describe another circle cutting the former circle at P and Q

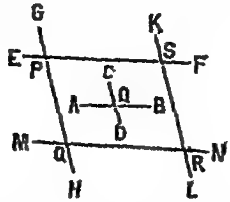
Locus of the points 4 cm distant from A is the circumference of a circle whose centre is A and radius=4 cm.

Locus of the points 5 cm distant from B is the circumference of a circle whose centre is B and radius=5 cm

Hence the points common to the two circles drawn with the centres A,B, and radii 4 cm., and 5 cm. *respectively*, must satisfy both the conditions, that is, P,Q, the points of intersection of two circles, will each be 4 cm. distant from A, and 5 cm. distant from B.

Q. E. F.

6. Let AB, CD be two given straight lines.



It is required to find points 3 cm. distant from AB, and 4 cm. from CD.

Let AB, CD cut one another at O. Draw EF and MN parallels to AB, each at a distance of 3 cm. from AB. Draw GH and KL parallels to CD, each at a distance of 4 cm. from CD, cutting EF at P and S, and MN at Q and R.

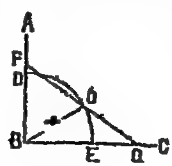
Locus of the points 3 cm. distant from AB is the pair of straight lines EF and MN drawn parallel to AB on either side of it, and at a distance of 3 cm from it.

Locus of the points 4 cm distant from CD is the pair of straight lines GH and KL drawn parallel to CD on either side of it, and at a distance of 4 cm. from it.

Hence the points common to the two pair of straight lines must satisfy both the conditions; that is, P, Q, R, S, the points of intersection of the two pair of straight lines, will each be 3 cm. distant from AB and 4 cm. distant from CD

Q. E. F.

7. Let AB, BC be two straight rulers placed at right angles to one another and let PQ be one position of the straight rod of given length which slides between them.



It is required to plot the locus of the middle point of PQ, and to show that this locus is a fourth part of the circumference of a circle.

Bisect PQ at O. Join BO. With the centre B and radius BO draw an arc cutting AB in D and BC in E.

BO $= \frac{1}{2}$ PQ (proved in Ex 10 on page 47)

= constant ($\because$ the straight rod is of given length)

i e the distance of O from B is always constant. But B is a fixed point, therefore the locus of O, the middle point of PQ, is a circle whose centre is B and radius $= \frac{1}{2}$PQ.

Because the rod PQ slides between the rulers AB and BC, therefore its middle point O will never go beyond them. Hence the arc DOE is the required locus

In a circle a radius starting from any position in any direction moves through 4 rt $\angle^s$ about the centre to come back to its original position (from which it started).

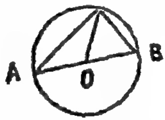
Now 1 rt. angle is a fourth part of 4 rt angles

$\therefore$ the arc DOE subtended by the rt $\angle$ DBE is a fourth part of the circumference of the circle drawn with centre B and radius BO.

Hence the required locus is the arc DOE which is a fourth part of the circumference of a circle.

Q. E. F

8. Let APB be a right-angled triangle described on the given base AB as a hypotenuse.



It is required to find the locus of its vertex P.

Bisect AB at O, and join PO.

Then PO $= \frac{1}{2}$ AB (proved in Ex 10 on page 47)

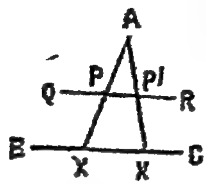
Because AB is given, hence its middle point O is a fixed point.

$\therefore$ the locus of P is a circle whose centre is O and radius $= \frac{1}{2}$ AB.

∴ the locus of vertices of right angled triangles discribed on the given base AB as a hypotenuse is a circle on AB as diameter.

Q. E. F.

9. Let A be a fixed point, and BC a fixed straight line on which a point X moves.



It is required to plot the locus of P, the middle point of AX, and prove that the locus is a straight line parallel to BC.

Let X, X′ be any two positions of X. Join AX and AX′

Bisect AX at P and AX′ at P′. Join PP′ and produce it beyond both ends to points Q and R

Because P and P′ are the middle points of AX and AX′

∴ the straight line PP′ is parallel to XX′ or BC (proved in Ex. 2 on page 64)

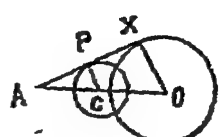
If any point in BC representing the position of X be joined to A, then this straight line will be bisected by YR or PP′ (proved in Ex 5 on page 64)

Thus it is evident that every straight line drawn from A and terminated by BC is bisected by QR which is parallel to BC

∴ QR is the required locus and it is parallel to BC.

Q E F.

10. Let A be a fixed point, and O the centre of a given circle on the circumference of which the point X moves.



It is required to plot the locus of P, the middle of AX, and to prove that this locus is a circle.

Join AO and bisect it at C.

Let X denote any position of the moving point on the circumference of the given circle.

Join OX and AX Bisect AX at P. Join CP.

Because P and C are the middle points of AX and AO.

$\therefore$ PC is $\frac{1}{2}$ of OX (proved in Ex. 3 on page 64)

Because O and A are fixed points, hence C the middle point of AO is also a fixed point.

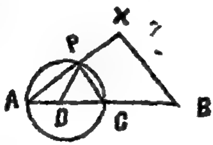
Again, because the circle with centre O is given, therefore the radius OX is of constant length.

$\therefore$ $\frac{1}{2}$ OX or CP is also of constant length.

$\therefore$ the locus of P, the middle point of AX, is a circle whose centre is C and radius $=\frac{1}{2}$ OX.

Q. E. F.

11. Let AB be a given straight line, and let AX be a perpendicular drawn from A to any straight line BX passing through B.



If BX revolve about B, it is required to find the locus of the middle point of AX.

Bisect AX at P and AB at C.

Join PC. Then PC is parallel to BX (proved in Ex. 2 on page 64)

Since BX and PC are parallel, and AX meets them

$\therefore$ the $\angle$ AXB $=$ the $\angle$ APC (Theor. 14)

But the $\angle$ AXB $=90^\circ$ (given)

$\therefore$ the $\angle$ APC is 90°.

$\therefore$ the circle described on the diameter AC will pass through P (Prop. 10, also proved in Ex 8)

Because AB is a fixed straight line and C is its middle point, therefore AC is of constant length

$\therefore$ the circle described on the diameter AC is a fixed circle.

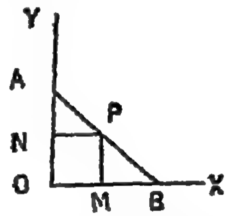
And because P, the middle point of AX, lies on this circle,

$\therefore$ the locus of P is the circumference of this circle.

Q. E. F.

12 Let OX and OY be two straight lines cutting at right angles at O, and let P be a point within the angle XOY from which perpendiculars PM, PN are drawn to OX, OY respectively.

(*i*). It is required to plot the locus of P when PM+PN =6 cm.



To find a position of P measure off along OX a length OM less than 6 cm. At M draw MP perpendicular to OX and equal to the difference between 6 cm. and OM. From P draw PN perpendicular to OY.

Then PM+PN=PM+OM=6 cm.

Here the point P moves through all positions in which PM+PN=6 cm.; hence one position of the moving point P is at the point B in OX, such that OB=6 cm. In this case PM=0 zero and PN coincides with BO.

Let P be any other position of the moving point. Draw PM, PN perpendiculars to OX, OY respectively.

Then $PM+PN=6$ cm.

Because $PN=OM$ (Theor. 21)

$\therefore PM+PN=PM+OM=6$ cm

Join BP and produce it to meet OY in A.

Because $OB=OM+MB=6$ cm.

and $OM+PM=6$ cm.

$\therefore OM+MB=OM+PM$

$\therefore MB=PM$

$\therefore$ the $\angle MBP=$ the $\angle MPB$ (Theor. 5)

Again because the $\angle PMB$ is a rt. $\angle$, therefore each of the $\angle^s$ MBP and MPB is half a rt angle (Theor. 16 Inf 3)

Thus B being a fixed point, and $\angle MBP$ a constant angle, the line BP is known in position

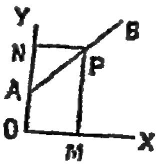
$\therefore$ Every point P which moves as given above, lies on the straight line BP which passes through the point B and makes with OX an angle equal to half a right angle.

But since the point P remains within the $\angle XOY$, therefore the line AB between the given lines OX, OY is the required locus.

Q. E F.

(ii) It is required to plot the locus of P when $PM-PN=3$ cm.

Here the point P moves through all positions in which $PM-PN=3$ cm., hence one position of the moving point P is at the point A in OY, such that $OA=3$ cm In this case $PN=0$ zero and MP coincides with OA.



Let P be any other position of the moving point.

Draw PM, PN perpendiculars to OX, OY respectively.

Then $PM-PN=3$ cm.

Join AP and produce it beyond P to any point B.

Because $OA = ON - NA = 3$ cm.

and $PM - PN = 3$ cm.

$\therefore ON - NA = PM - PN$

But $ON = PM$. (Theor. 21)

$\therefore NA = PN$

$\therefore$ the $\angle NAP =$ the $\angle NPA$ (Theor. 5)

Again because the $\angle ANP$ is a right angle, therefore each of the $\angle^s$ NAP and NPA is half a rt. angle. (Theor. 16 Inf. 3)

Thus A being a fixed a point, and $\angle NAP$ a constant angle, the line AP is known in position.

$\therefore$ every point P which moves as given above, lies on the straight line AP which passes through the point A and makes with OY an angle equal to half a rt. angle.

$\therefore$ the line AB is the required locus.

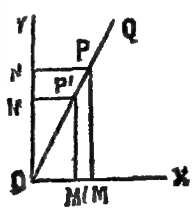
Thus the locus of P is the straight line AB lying between the given straight lines OX, OY and making with OY an angle equal to half a right angle.

Q. E. F.

13. Let two straight lines OX, OY intersect at right angles at O and let P be the moving point from which perpendiculars PM, PN are drawn to OX, OY.

(i) It is required to plot (without proof) the locus of P, when $PM = 2\,PN$.

Take any point M′ in OX. From OY cut off $ON' = 2 \cdot OM'$. At M′ draw M′P′ perpendicular to OX. At N′ draw N′P′ perpendicular to OY, meeting M′P′ in P′.



Take another point M in OX. From OY cut off ON = 2 OM.

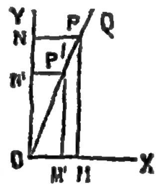
At M draw MP perpendicular to OX. At N draw NP perpendicular to OY, meeting MP in P. Join P′P and produce it to any point Q.

Then the line PP′ is the required locus.

It should be noticed that this locus passes through O.

Q. E. F

(ii) It is required to plot (without proof) the locus of P when PM = 3 PN.



Take any point M′ in OX. From OY cut off ON′ = 3 OM′. At M′, N′ draw M′P′, N′P′ perpendiculars to OX, OY respectively, the lines meeting in P′.

Take another point M in OX. From OY cut off ON = 3 OM. At M, N draw MP, NP perpendiculars to OX, OY respectively, the lines meeting in P.

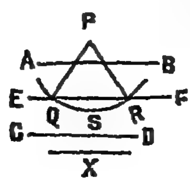
Join P′P and produce it to any point Q.

Then the line PP′ is the required locus.

It should be noticed that this locus passes through O.

Q. E. F

14. Let AB, CD be two given parallel straight lines, X a given line, and P a given point.



It is required to find a point which is at a given distance X from P, and is equidistant from two given parallel straight lines AB, CD

Locus of the points equidistant from AB and CD is the straight line EF parallel to AB or CD, and midway between them.

Locus of the points which are at a given distance X from P is the circumference of a circle QR whose centre is P and radius equal to the given distance X.

Hence the points common to the circle QR and the straight line EF must satisfy both the conditions: that is, Q, R the points of intersection of the circle QR and the straight line EF are equidistant from AB and CD, and are at the given distance X from P.

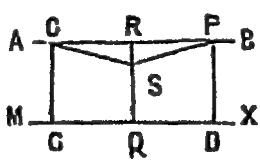
Q. E. F.

This problem admits of two solutions when the given distance X is greater than the perpendicular from P to EF).

This problem admits of only one solution when the given distance X is equal to the distance of P from the straight line EF.

This problem is impossible when the given distance X is less than the distance of P from the straight line EF.

15. Let MX be a given straight line, and let S be a fixed point 2″ distant from it.



It is required to find two points which are $2\frac{3}{4}''$ distant from S, and also $2\frac{3}{4}''$ distant from MX.

From S draw SQ perpendicular to MX and produce QS to any point R, making $RQ = 2\frac{3}{4}''$.

Through R draw ARB parallel to MX.

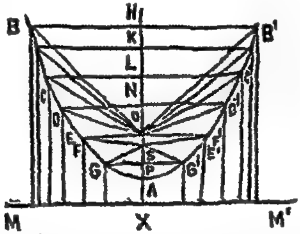
With the centre S and radius $=2\frac{3}{4}''$ draw an arc cutting AB at the points O and P

Join SO and SP. From O and P draw OC and PD perpendiculars to MX

Then O and P are the two points which are $2\frac{3}{4}''$ distant from S, and also $2\frac{3}{4}''$ distant from MX ($\because$ $SO=2\frac{3}{4}''=QR=OC$)

Q. E. F.

16. Let MX be a given straight line and S a given point.



It is required to find a series of points equidistant from the given point S and the given straight line MX, and to draw a curve (freehand) passing through all the points so found.

From S draw SX perpendicular to MX and bisect it at A.

Then A is a point equidistant from S and MX.

Produce MX beyond X to any point M′.

Produce XS beyond S and take several points H, K, L, N, O, P, ... on it.

Through H, K, L, N, O, P,......draw parallels to MXM′.

With the centre S and radius=HX draw an arc cutting the parallel through H at the points B, B′.

With the centre S and radius=KX draw an arc cutting the parallel through K at C, C′.

With the centre S and radius=LX draw an arc cutting the parallel through L at D D′,

With the centre S and radius=NX draw an arc cutting the parallel through N at E, E′.

With the centre S and radius=OX draw an arc cutting the parallel through O at F.F′.

With the centre S and radius=PX draw an arc cutting the parallel through P at G,G′

Draw a freehand curve passing through the points B, C, D, E, F, G, A, G′, F′, E′, D′, C′, B′ thus found as shown in the figure.

The curve so formed is called a **Parabola.**

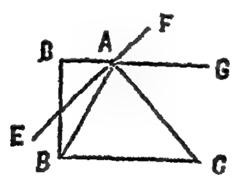
Join B S, B′S, CS, C′S, DS, D′S, ES, E′S, FS, F′S, GS, G′S

From B B′ C,C′, D,D′, E,E′, F,F′, G,G′ draw perpendiculars to MXM′.

Then B,B′, C,C′, D,D′, E,E′, FF′, G G′ are the series of points equidistant from S and MX (proved in Ex. 15 on page 95).

Q. E. F.

17. Let EF be a given straight line.



It is required to contruct a triangle of given altitude on a given base having its vertex on the given straight line EF.

Let BC represent the given base of the triangle. At B draw BD perpendicular to BC making it equal to the given altitude. From D draw DG parallel to BC.

Then the vertex of the triangle lies on DG.

Also the vertex lies on the straight line EF (given)

$\therefore$ the point A where DG cuts EF is the required vertex.

Join AB and AC

Then ABC is the required triangle.

Q E. F.

18 (See figure in Ex. 7 on page 34).

Let ABC be a triangle.

It is required to find a point equidistant from the three sides of the triangle ABC.

Bisect the $\angle^s$ABC and ACB by BO and CO, the lines BO and CO meeting in O.

Locus of the points equidistant from CB and CA is the line CO which bisects the $\angle$ACB. (Prob. 15)

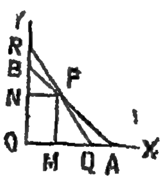
Locus of the points equidistant from BA and BC is the line BO which bisects the $\angle$ABC. (Prob 15)

Hence the point O where these two bisectors meet is equidistant from AB, AC and BC

Q. E F.

19 Let two straight lines OX, OY cut at right angles at O, and let Q, R be two points in OX and OY respectively.

(i) It is required to plot the locus of the middle point of QR when OQ+OR=constant.



Join QR and bisect it at P.

Here the line QR moves through all positions in which OQ+OR is constant, and P is its middle point. Therefore one position of P, when QR falls along OX, is at A in OX.

When QR (while moving) falls along OX in such a way that $OQ+OR$ is constant, there is a line in OX which represents the position of QR. This line is equal to $OQ+OR$.

But A representing the position of P, the middle point of QR, is the middle point of this line.

$\therefore$ $OA=\frac{1}{2}(OQ+OR)$=constant.

Hence A is a fixed point.

From P draw PM perpendicular to OX, and PN perpendicular to OY.

Join AP and produce it beyond P to meet OY in B.

In the $\triangle$ ROQ, P is the middle point of RQ, and PM is parallel to OR (proved in Ex. 2 on page 41)

$\therefore$ M is the middle point of OQ (proved in Ex. 1 on page 64)

Since P, M are the middle points of RQ, OQ

$\therefore$ $PM=\frac{1}{2}$ of RO (proved in Ex. 3 on page 64)

Similarly, $PN=\frac{1}{2}$ of OQ

$\therefore$ $PM+PN=\frac{1}{2}(RO+OQ)=OA$

But $OA=OM+MA$, and $PN=OM$ (Theor. 21)

$\therefore$ $PM+PN=OM+MA=PN+MA$

$\therefore$ $PM=MA$

$\therefore$ the $\angle$ MPA = the $\angle$ MAP (Theor. 5)

But the $\angle$ AMP is a rt. angle, hence each of the $\angle^s$ MPA, MAP is half a rt. angle.

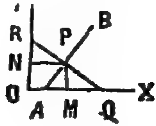
Thus A being a fixed point, and $\angle$ MAP a constant angle, AB is a fixed line.

$\therefore$ In every position of QR the middle point P lies on the straight line APB which makes with OX an angle equal to half a rt. angle.

But since the points Q, R always lie on OX and OY, therefore AB lying between OX and OY is the required locus as shown in the figure.

Q. E. F.

(*ii*) It is required to plot the locus of the middle point of QR when OQ − OR is constant.



Join QR and bisect it at P.

Here the line QR moves through all positions in which OQ − OR is constant, and P is its middle point. Therefore one position of P when QR falls along OX, is at A in OX.

When QR (while moving) falls along OX in such a way that OQ − OR is constant, there is a line in OX which represents the position of QR This line is equal to OQ − OR.

But A representing the position of P, the middle point of QR, is the middle point of this line.

$\therefore$ OA $= \frac{1}{2}$ (OQ − OR) = constant.

Hence A is a fixed point.

From P draw PM perpendicular to OX, and PN perpendicular to OY.

Join AP and produce it beyond P to any point B

In the $\triangle$ ROQ P is the middle point of RQ, and PM, RO are parallel (proved in Ex 2 on page 41).

$\therefore$ M is the middle point of OQ (proved in Ex. 1 on page 64)

Since P, M are middle points of RQ, OQ,

$\therefore$ PM is $\frac{1}{2}$ of RO (proved in Ex 3 on page 64)

Similarly, PN $= \frac{1}{2}$ of OQ

$\therefore$ PN − PM $= \frac{1}{2}$ (OQ − OR) = OA.

But OA = OM − AM, and OM = PN

$\therefore$ PN − PM = OM − AM = PN − AM

$\therefore$ PM = AM

∴ the ∠ MPA = the ∠ MAP (Theor. 5)

But the ∠ AMP is a rt angle, hence each of the ∠s MPA, MAP is half a rt. angle.

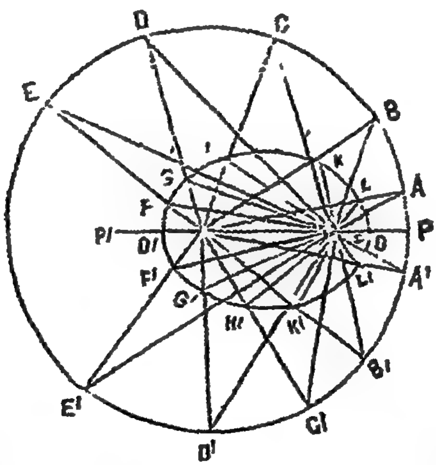
Thus A being a fixed point, and the ∠ MAP a constant angle, AB is a fixed line

∴ in every position of QR, the middle point P lies on the straight line APB which makes with OX an angle equal to half a rt. angle.

But since the points Q,R always lie on OX and OY therefore AB lying between OX and OY is the required locus as shown in the figure.

20 (i) Let S, S′ be two fixed points.

It is required to find a series of points P such that SP+, S′P = 3·5″.



With the centre S and radius = 3·5″ draw a circle.

Join SS′ and produce it beyond S′ to meet the circumference of the circle in P. Bisect S′P at O. Then O is a point on the curve.

Produce S′S beyond S to a point P′ making SP′ = S′P. Bisect SP′ at O′. Then O′ is another point on the curve.

Take any number of points E,D,C,B,A,A′,B′,C′,D′,E′,...... on the circumference of the circle.

Join SE, S′E, SD, S′D,SC,S′C,SB,S′B,SA,S′A,SA′,S′A′, SB′, S′B,′SC,′S′C,′SD,′S′D,′SE′S′E,′......

At S′ make the ∠ ES′F=the ∠ SES′ the arm S′F meeting SE in F.

At S′ make the ∠ DS′G=the ∠ SDS′ the arm S′G meeting SD in G.

At S′ make the ∠ CS′H=the ∠ SCS′ the arm S′H meeting SC in H

At S′ make the ∠ BS′K=the ∠ SBS′ the arm S′K meeing SB in K.

At S′ make the ∠ AS′L=the ∠ SAS′ the arm S′L meeting SA in L.

At S′ make the ∠ A′S′L′=the ∠ SA′S′ the arm S′L′ meeting SA′ in L′.

At S′ make the ∠ B′S′K′=the ∠ SB′S′ the arm S′K′ meeting SB′ in K′.

At S′ make the ∠ C′S′H′=the ∠ SC′S′ the arm S′K′ meeting SC′ in H′.

At S′ make the ∠ D′S′G′=the ∠ SD′S′ the arm S′G′ meeting SD′ in G′.

At S′ make the ∠ E′S′F′=the ∠ SE′S′ the arm S′F′ meeting SE′ in F′.

Draw a freehand curve passing through the points F,G, H,K,L,O,L′,K′,H′,G′,F′,O′ thus found as shown in the figure.

The curve so formed is called an Ellipse.

S′O=OP, SO′=O′P′ and S′P=SP′

∴ SO+S′O=SO+OP=SP=3·5″

Also S′O′+SO′=SS′+SO′+O′P′=SS′+SP′=SS′+S′P =SP=3 5″

∴ O and O′ are the points on the locus.

Because the $\angle$ S′AL = the $\angle$ AS′L

$\therefore$ AL = S′L (Theor. 6)

$\therefore$ SL + S′L = SL + LA = SA = 3·5″

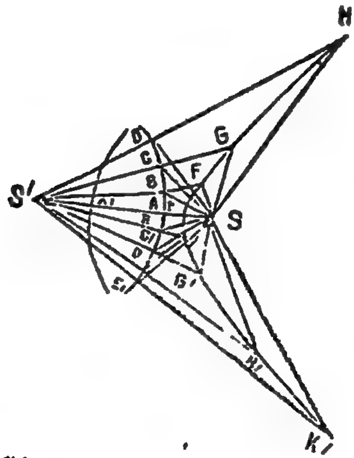
$\therefore$ L is a point on the locus.

Similarly, it can be proved that K, H, G, F, F′, G′, H′, K′, L′ are points on the locus.

Q. E. F.

(*ii*) Let S, S′ be two fixed points.

It is required to find a series of points P such that SP — S′P = 1 5″.



With the centre S′ and radius = 1·5″ draw an arc.

Join SS′ cutting the arc at A, bisect AS, then the middle point lies on the curve.

Take any number of points D, C, B, B′, C′, D′, E′,......

Join S′D SD, S′C, SC, S′B, SB, S′B′, SB′, S′C′, SC′, S′D′, SD′, S′E′, SE′,.........

Produce S′D beyond D, S′C beyond C, S′B beyond B, S′B′ beyond B′, S′C′ beyond C′, S′D′ beyond D′, S′E′ beyond E′,......

At S make the $\angle$ DSH = the $\angle$ SDH the arm SH meeting S′D produced in H.

At S make the $\angle$ CSG = the $\angle$ SCG the arm SG meeting S′C produced in G.

At S make the $\angle$ BSF = the $\angle$ SBF the arm SF meeting S′B produced in F

At S make the $\angle$ B′SF′ = the $\angle$ SB′F′ the arm SF′ meeting S′B′ produced in F′.

At S make the $\angle$ C′SG′ = the $\angle$ SC′G′ the arm SG′, meeting S′C′ produced in G′.

At S make the $\angle$ D′SH′ = the $\angle$ SD′H′ the arm SH′ meeting S′D′ produced in H′.

At S make the $\angle$ E′SK′ = the $\angle$ SE′K′ the arm SK′ meeting S′E′ produced in K′.

Draw a freehand curve passing through the points H, G, F, the middle point of AS, F′, G′, H′, K′, thus found as shown in the figure.

Similarly, with centre S and radius = 1·5″ draw an arc cutting SS′ at A′. By taking any number of points on this arc, and determining the corresponding points on the locus (by the method given above) a similar curve can be drawn on the other side of AA′

These *two* branches of the so formed curve form what is called a **Hyperbola**

Suppose P represents the middle point of AS,

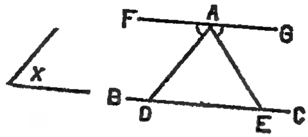
Then S′P − SP = S′P − AP = S′A = 1 5″

$\therefore$ P is a point on the curve
Because the $\angle$ FBS = the $\angle$ FSB
$\therefore$ FB = FS (Theor. 6)
$\therefore$ S′F − SF = S′F − FB = S′B = 1·5″
$\therefore$ F is a point on the locus.
Similarly, it can be proved that H, G, F′, G′, H′, K′, re points on the locus

Q. E. F.

Page 98.

1. Let A be a given point, X a given angle and BC a given straight line.



It is required to draw a straight line from A to make with BC an angle equal to the given angle X.

Construction—Through A draw a straight line FAG parallel to BC. At A make the angles FAD and GAE each equal to the given angle X, the arms AD and AE meeting BC in D and E.

Then AD and AE are the required lines.

Proof—Because FA and DC are parallel, and AD meets them

∴ the ∠ FAD = the alternate ∠ ADC (Theor. 14)

But the ∠ FAD = the ∠ X, ∴ the ∠ ADC = the ∠ X.

Again, because GA and EB are parallel, and AE meets them

∴ the ∠ GAE = the alternate ∠ AEB (Theor. 14)

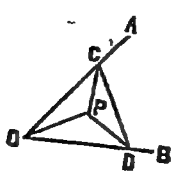
But the ∠ GAE = the ∠ X; ∴ the ∠ AEB = the ∠ X,

∴ AD and AE are the required lines.

Two lines can be drawn from the given point A making with BC angles each equal to the given angle X.

Q. E. F.

2. Let AOB be a given angle.



It is required to draw the bisector of the angle AOB, without using the vertex O in the construction.

Construction —Take any point C in OA, and any point D in OB Join CD.

Bisect the $\angle^s$ ODC and OCD by DP, CP, the arms DP, CP meeting in P. Join PO.

Then PO is the bisector of the $\angle$ AOB.

Proof.—Because CP bisects the $\angle$ OCD

$\therefore$ CP is the locus of points equidistant from CO and CD (Prob. 15)

Again, because DP bisects the $\angle$ ODC

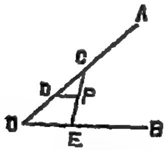
$\therefore$ DP is the locus of points equidistant from DO and DC (Prob. 15)

$\therefore$ P is on the locus of points equidistant from OC and OD (proved in Prop. II on page 96)

That is, OP is the bisector of the $\angle$ COD or AOB

Q. E. F.

3 Let P be a given point within the $\angle$ AOB.



It is required to draw through P a straight line terminated by OA, and OB, and bisected at P

Construction.—From P draw PD parallel to OB meeting OA in D.

From DA cut off DC = OD. Join CP and produce it to meet OB in E

Then CE is the required straight line.

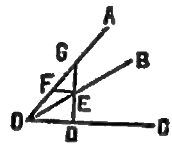
Proof —Because D is the middle point of OC (by consction) and DP is parallel to OE.

∴ P is the middle point of CE (proved in Ex. 1 on page 64).

Thus, through P the straight line CE is drawn terminated by OA and OB, and bisected at P.

Q. E. F.

4. Let OA, OB, OC be three straight lines meeting at O.



It is required to draw a transversal terminated by OA and OC and bisected by OB.

Construction.—Take any point E in OB.

From E draw EF parallel to OC meeting OA in F.

From FA cut off FG=OF

Join GE and produce it to meet OC in D.

Then GD is the required straight line.

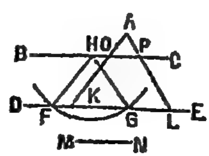
Proof.—Because F is the middle point of OG (by construction) and FE is parallel to OD

∴ E is the middle point of GD (proved in Ex. 1 on page 64)

Thus GD is the transversal terminated by OA and OC, and bisected by OB at E

Q. E. F.

5. Let MN be a given straight line, A a given point and BC, DE two given parallel straight lines.



It is required to draw through the given point A a straight line so that the part intercepted between the two given parallels BC and DE may be of given length MN.

Construction —Take any point H in BC.

With the centre H and radius = MN draw an arc cutting DE at F and G. Join HF and HG.

From A draw AK parallel to HF cutting BC in O and meeting DE in K.

From A draw AL parallel to HG cutting BC in P and meeting DE in L.

Then AOK and APL are the required lines.

Proof—Because HO is parallel to FK (given), and HF is parallel to OK by construction)

$\therefore$ the figure HFKO is a parallelogram

$\therefore$ HF = OK (Theor. 21)

Again, because HG is parallel to PL (by construction) and HP is parallel to GL (given)

$\therefore$ the figure HGLP is a parallelogram

$\therefore$ HG = PL (Theor. 21)

But HF = HG = MN (by construction)

$\therefore$ OK = PL = MN.

$\therefore$ the straight lines AOK and APL have their parts OK and PL intercepted between the parallels BC and DE each equal to MN.

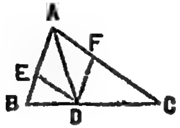
Q. E. F.

This problem admits of two solutions when the given length MN is greater than the distance of P from the line DE (*i.e.*, the perpendicular from P to DE), as shown in the figure.

This problem admits of only one solution when the given length MN is equal to the distance of P from the line DE.

This problem is impossible when the given length MN is less than the distance of P from the line DE.

6. Let ABC be a triangle.



It is required to inscribe a rhombus in the triangle ABC having one of its angles coinciding with the angle A.

Construction—Bisect the $\angle$ BAC by AD to meet BC in D

Through D draw DE parallel to AC meeting AB in E; also draw DF parallel to AB meeting AC in F.

Then AEDF is the required rhombus.

Proof—Because AF and ED are parallel, also AE and DF are parallel

$\therefore$ the figure AEDF is a parallelogram

$\therefore$ AE=DF, and AF=ED (Theor 21)

Because AF and ED are parallel and AD meets them

$\therefore$ the $\angle$ FAD=the alternate $\angle$ ADE (Theor 14).

But the $\angle$ EAD=the $\angle$ FAD (by construction)

$\therefore$ the $\angle$ ADE=the $\angle$ EAD

$\therefore$ ED=EA (Theor. 6)

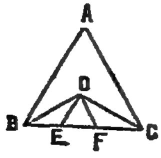
But AE=DF and AF=ED

$\therefore$ AE=DF=ED=AF.

$\therefore$ the figure AEDF is a rhombus.

Q. E. F.

7. Let BC be a given straight line.



It is required to use the properties of an equilateral triangle to trisect the given straight line BC.

Construction—With the centres B and C, and radius =BC draw two arcs cutting at A. Join AB and AC. Then ABC is an equilateral triangle.

$\therefore$ each of its angles ABC, ACB and BAC=60°. Bisect the $\angle$ ABC by BD; also bisect the $\angle$ ACB by CD, the bisector CD meeting DD in D.

From D draw DE parallel to AB to meet BC in E. From D draw DF parallel to AC to meet BC in F.

Then the line BC is trisected at E and F.

Proof—Because AB and DE are parallel, and BD meets them

$\therefore$ the $\angle$ BDE=the alternate $\angle$ ABD (Theor. 14)

But the $\angle$ ABD=the DBE (by construction)

$\therefore$ the $\angle$ DBE=the $\angle$ BDE

$\therefore$ BE=DE (Theor. 6)

Again, because DF and AC are parallel, and DC meets them

$\therefore$ the $\angle$ DCA=the alternate $\angle$ CDF (Theor. 14)

But the $\angle$ ACD=the $\angle$ DCF (by construction)

$\therefore$ the $\angle$ CDF=the $\angle$ DCF

$\therefore$ DF=CF (Theor. 6)

Again, because DE and AB are parallel, and BF meets them

$\therefore$ the $\angle$ DEF=the $\angle$ ABF (Theor. 14)=60°

Again, because DF and AC are parallel, and EC meets them

$\therefore$ the $\angle$ DFE=the $\angle$ ACE (Theor. 14)=60°

$\therefore$ the $\angle$ EDF=180°−(60°+60°) =180°−120°=60° (Theor 16 Inf 1)

$\therefore$ DEF is an equilateral triangle (Cor. Theor. 6)

∴ DE=EF=DF

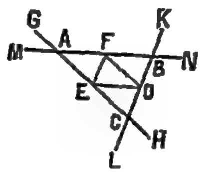
But BE=DE and CF=DF (proved)

∴ BE=EF=FC.

That is, BC is trisected at E and F.

Q E. F.

8 (i) Let D, E, F be the middle points of the three sides of a triangle.



It is required to construct the triangle.

Construction—Join DE, EF and FD.

Through D draw LDK parallel to EF.

Through E draw HEG parallel to DF cutting LK at C.

Through F draw MFN parallel to DE cutting LK at B and HG at A.

Then ABC is the required triangle.

Proof—Because DB is parallel to EF, and DE is parallel to BF

∴ DEFB is a parallelogram

∴ DB=EF (Theor 21)

Similarly, CDFE is a parallelogram

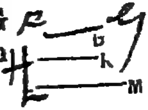
∴ CD=EF (Theor 21)

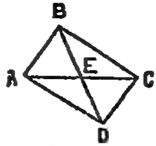
∴ CD=DB, *i.e.*, D is the middle point of BC

Similarly, it can be proved that E is the middle point of AC and F is the middle point of AB.

Q. E F.

(*ii*) Let HK and LM denote the lengths of two sides of a triangle, and FG the length of the median, which bisects the third side.





It is required to construct the triangle

Construction—Take a straight line BD equal to 2 FG. With centres B and D, and radii equal to HK and LM respectively draw two arcs cutting at A. Join AB

Bisect BD at E Join AE and produce it to C making EC = AE Join BC

Then ABC is the required triangle.

Join AD.

Proof—In the two $\triangle^s$ AED and BEC

Because $\begin{cases} AE = EC \text{ (by construction)} \\ DE = EB \text{ (by construction)} \\ \text{and the } \angle AED = \text{the } \angle BEC \text{ (Theor. 3)} \end{cases}$

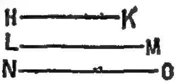
$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

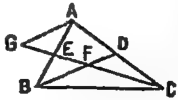
so that, AD = BC = LM

AB = HK, and BE = $\frac{1}{2}$ BD = FG.

Q E. F.

(*iii*) Let NO denote the length of one side of a triangle, and HK, LM the lengths of the medians which bisect the other two sides.





It is required to construct a triangle

Construction—Take a straight line BC = NO.

With the centres B and C, and radii equal to $\frac{2}{3}$ HK and $\frac{2}{3}$ LM respectively draw two arcs cutting at F. Join BF and FC.

Produce CF to E making $FE = \frac{1}{2} FC$. Join BE and produce it to A making $EA = BE$. Join AC.

Then ABC is the required triangle.

Produce BF to meet AC in D. Also produce FE to G making $EG = EF$. Join GA.

Proof—In the two $\triangle^s$ AEG and BEF

$$\text{Because} \begin{cases} AE = BE \text{ (by construction)} \\ GE = EF \text{ (by construction)} \\ \text{and the } \angle AEG = \text{the } \angle BEF \text{ (Theor 3)} \end{cases}$$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, $AG = FB$, and the $\angle GAE =$ the $\angle EBF$

But the $\angle^s$ GAE and EBF are alternate angles

$\therefore$ GA and FB are parallel (Theor. 13)

Because $EF = EG$ and $EF = \frac{1}{2} FC$ (by construction)

$\therefore EF + EG$ or $FG = FC$

Now, in the $\triangle$ AGC, F is the middle point of GC, and FD is parallel to GA

$\therefore$ D is the middle of AC (proved in Ex. 1 on page 64).

Because F and D are the middle points of GC and AC

$\therefore FD = \frac{1}{2} GA$ (proved in Ex. 3 on page 64) $= \frac{1}{2} BF$

$FC = \frac{2}{3} LM$, and $EF = \frac{1}{2} FC$

$\therefore EF = \frac{1}{3} LM$.

$\therefore EC = CF + FE = \frac{2}{3} LM + \frac{1}{3} LM = LM$.

Also $BF = \frac{2}{3} HK$, and $FD = \frac{1}{2} BF$

$\therefore FD = \frac{1}{3} HK$

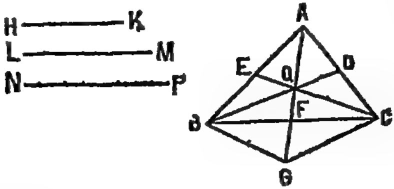
$\therefore BD = BF + FD = \frac{2}{3} HK + \frac{1}{3} HK = HK$.

$\therefore$ BD and EC are the medians to AC and AB and their lengths are equal to HK and LM respectively.

$\therefore$ ABC is the required triangle.

Q. E. F.

(*iv*) Let HK, LM, NP be the lengths of the three medians of a triangle.



It is required to construct the triangle.

Construction—Take a straight line $AO=\frac{2}{3}$ HK, and produce it to G making $OG=AO$.

With centres O and G, and radii equal to $\frac{2}{3}$ NP and $\frac{2}{3}$ LM respectively draw two arcs cutting at B Join BO and BG.

Bisect OG at F Join BF and produce it to C making $FC=BF$ Join AB and AC.

Then ABC is the required triangle

Join CO and produce it to meet AB in E.

Produce BO to meet AC in D Join GC.

Proof.—Because $AO=\frac{2}{3}$ HK, and $OF=\frac{1}{2}\,OG=\frac{1}{2}\,AO$

$\therefore\ OF=\frac{1}{3}$ HK

$\therefore\ AF=AO+OF=\frac{2}{3}\,HK+\frac{1}{3}\,HK=HK$

also F is the middle point of BC (by construction)

$\therefore$ AF is the median to BC.

Now in the two Δ^s OBF and FGC

$$\text{Because}\begin{cases}OF=FG \text{ (by construction)}\\ BF=FC \text{ (by construction)}\\ \text{and the } \angle OFB=\text{the } \angle GFC \quad \text{(Theor. 3)}\end{cases}$$

$\therefore$ two Δ^s are equal in all respects (Theor. 4)

so that, $BO=CG$, and the $\angle$ BOF = the $\angle$ FGC

But the $\angle^s$ BOF and FGC are alternate angles

$\therefore$ BO and CG are parallel (Theor. 13)

Again, in the two Δ^s OFC and BFG

$$\text{Because}\begin{cases}OF=FG \text{ (by construction)}\\ FC=BF \text{ (by construction)}\\ \text{and the } \angle OFC=\text{the } \angle BFG \text{ (Theor. 3)}\end{cases}$$

$\therefore$ two $\triangle^s$ are equal in all respects (Theor. 4)

so that, $OC=BG$ and the $\angle FOC=$the $\angle BGF$.

But the $\angle^s$ FOC and BGF are alternate angles

$\therefore$ OC and BG are parallel (Theor. 13)

In the $\triangle$ AGC, O is the middle point of AG (by construction), and OD is parallel to GC (proved)

$\therefore$ D is the middle point of AC (proved in Ex. 1 on page 64)

Since O and D are the middle points of AG and AC

$\therefore OD=\frac{1}{2}$ of GC (proved in Ex. 3 on page 64)

But $GC=BO$ (proved), therefore $OD=\frac{1}{2}$ of BO.

Because $BO=\frac{2}{3}$ NP and $OD=\frac{1}{2}$ BO

$\therefore OD=\frac{1}{3}$ NP

$\therefore BD=BO+OD=\frac{2}{3}NP+\frac{1}{3}NP=NP$

and D is the middle point of AC (proved)

$\therefore$ BD is the median to AC.

In the $\triangle$ ABG, O is the middle point of AG (by contruction), and OE is parallel by BG (proved)

$\therefore$ E is the middle point of AB (proved in Ex. 1 on page 64)

Since O and E are the middle points of AG and AB

$\therefore OE=\frac{1}{2}$ of BG (proved in Ex. 3 on page 64)

But $BG=OC$ (proved), therefore $OE=\frac{1}{2}$ of OC

$BG=\frac{2}{3}$ LM (by construction), therefore $OC=\frac{2}{3}$ LM

Because $CO=\frac{2}{3}$ LM, and $OE=\frac{1}{2}$ OC

$\therefore$ $OE = \frac{1}{3} LM$

$\therefore$ $CE = CO + OE = \frac{2}{3}LM + \frac{1}{3} LM = LM$

also E is the middle point of AB (proved)

$\therefore$ CE is the median to AB.

Q. E. F.